# Algebraic Solution of the Isovector Pairing Problem

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**Abstract**

A simple and effective algebraic isospin projection procedure for constructing basis vectors of irreducible representations of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ from those in the canonical $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ basis is outlined, which is useful in dealing with the isovector pairing problem. The expansion coefficients are components of null-space vectors of the projection matrix. Explicit formulae for evaluating $O_T(3)$-reduced matrix elements of $O(5)$ generators are derived.

**Keywords:** *Isovector pairing; proton-neutron quasi-spin group; dynamical symmetry*

## 1 Introduction

The proton-neutron quasi-spin group generated by an $O(5)$ algebra is very useful in dealing with nucleon pairing problems in a shell model framework [1–5]. Due to its importance in nuclear spectroscopy, irreducible representations (irreps) of $O(5)$ have been studied in various ways. The most natural basis for irreps of $O(5)$ may be the branching multiplicity-free canonical one with $O(5) \supset O(4)$, where $O(4)$ is locally isomorphic to $SU_\Lambda(2) \otimes SU_I(2)$, of which the construction of the basis vectors was presented in Refs. [6–8]. The matrix representations of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ were provided in Refs. [6–9]. Since the isospin is approximately conserved in the charge-independent isovector pairing problem, it is more convenient to adopt the non-canonical $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis for this case, where $O_T(3)$ is the isospin group, and $O_{\mathcal{N}}(2) \sim U_{\mathcal{N}}(1)$ is related with the number of nucleons in the system. The main problem is that the reduction $O(5) \downarrow O_T(3) \otimes O_{\mathcal{N}}(2)$ is no longer branching multiplicity-free in general. Basis vectors of $O(5)$ irreps in the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis can be either expanded in terms of those in the $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ or constructed by using tensor coupling methods directly, for which various attempts were made [6, 10–15]. A recent survey on the subject with relevant references is provided in Refs. [16, 17]. Though various procedures for the construction of basis vectors of $O(5)$ irreps in the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ were provided in these works, only cases up to the branching multiplicity three were obtained explicitly

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*http://www.ntse.khb.ru/files/uploads/2018/proceedings/Pan.pdf.*

in the past. Moreover, though there are closed expressions of the expansion coefficients (overlaps) [13] of the basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ in terms of those of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ for any irrep of $O(5)$, a triple sum is involved. Especially, the basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ obtained in all previous works [6, 10–15] are non-orthogonal with respect to the branching multiplicity label, of which direct computation will be CPU time consuming.

## 2 $O(5)$ in the $O_T(3) \otimes O_{\mathcal{N}}(2)$ basis

The generators of $O(5)$ in the $O(5) \supset O_T(3) \times O_{\mathcal{N}}(2)$ basis may be expressed as

$$\begin{aligned} &A_1^\dagger = \nu_+, \quad A_{-1}^\dagger = \tau_+, \quad A_1 = \nu_-, \quad A_{-1} = \tau_-, \\ &A_0^\dagger = U_{\frac{1}{2}\frac{1}{2}}, \quad A_0 = -U_{-\frac{1}{2}-\frac{1}{2}}, \quad T_+ = -\sqrt{2}\,U_{\frac{1}{2}-\frac{1}{2}}, \quad T_- = -\sqrt{2}\,U_{-\frac{1}{2}\frac{1}{2}}, \\ &T_0 = \nu_0 - \tau_0, \quad \hat{\mathcal{N}} = \nu_0 + \tau_0, \end{aligned} \tag{1}$$

where $\{T_+, T_-, T_0\}$ generate the subgroup $O_T(3)$, and $\hat{\mathcal{N}}$ generates the $O_{\mathcal{N}}(2)$. $\hat{\mathcal{N}} = \frac{\hat{n}}{2} - \Omega$, where $\Omega = \sum_j (j + 1/2)$ and the sum runs over all single-particle orbits considered, and $\hat{n}$ is the total number operator of valence nucleons, which is used in the isovector pairing model [1–5]. Moreover, $\{\nu_+ = A_1^\dagger,\ \nu_- = A_1,\ \nu_0 = \hat{n}_\pi/2 - \Omega/2\}$ and $\{\tau_+ = A_{-1}^\dagger,\ \tau_- = A_{-1},\ \tau_0 = \hat{n}_\nu/2 - \Omega/2\}$, where $\hat{n}_\pi$ and $\hat{n}_\nu$ are valence neutron and proton number operator, respectively, generate the $SU_\Lambda(2) \otimes SU_I(2)$ related to the quasispin of protons and neutrons with $\Lambda = (\Omega - v_\pi)/2$ and $I = (\Omega - v_\nu)/2$, where $v_\pi$ and $v_\nu$ are proton and neutron seniority numbers, respectively. The matrix elements of the double-tensor $\mathbf{U}$ introduced in Eq. (1) under the $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ basis were given in Refs. [6–9].

For a given irrep $(v_1, v_2)$ of $O(5)$, the basis vectors of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ are denoted as

$$\left| \begin{matrix} & (v_1, v_2) & \\ \Lambda = \frac{1}{2}(u_1 + u_2), & & I = \frac{1}{2}(u_1 - u_2) \\ m_\Lambda, & & m_I \end{matrix} \right\rangle, \tag{2}$$

where $m_\Lambda$ and $m_I$ are quantum number of $\nu_0$ and $\tau_0$, respectively, $u_1 = v_1 - q$ and $u_2 = v_2 - p$ with $p = 0, 1, \ldots, 2v_2$, and $q = 0, 1, \ldots, v_1 - v_2$.

As can be observed from Eq. (1), the basis vectors of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ given in Eq. (2) are also eigenstates of $T_0$ and $\hat{\mathcal{N}}$ with eigenvalues

$$M_T = m_\Lambda - m_I, \quad \mathcal{N} = m_\Lambda + m_I. \tag{3}$$

For a given irrep $(v_1, v_2)$ of $O(5)$, all possible basis vectors of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2) \supset U_\Lambda(1) \otimes U_I(1)$ shown in Eq. (2) restricted by the conditions (3) form a complete set for the fixed $M_T$ and $\mathcal{N}$. Therefore, the basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ can be expanded in terms of them with the restriction on the quantum numbers $m_\Lambda = \frac{1}{2}(\mathcal{N} + M_T)$ and $m_I = \frac{1}{2}(\mathcal{N} - M_T)$. In constructing the basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ for the irrep $(v_1, v_2)$ of O(5) with fixed $\mathcal{N}$, there is a freedom to choose a specific basis vector of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ with isospin $T$ and the quantum number of the third component of the isospin $M_T$. Practically, it is convenient to choose the highest or the lowest weight state of $O_T(3)$ with $M_T = T$ or $M_T = -T$. Here, we choose the highest weight state of $O_T(3)$ with $M_T = T$ as a reference state with

$$\left|\begin{matrix} (v_1, v_2) \\ \zeta\ T = M_T, \mathcal{N} \end{matrix}\right\rangle, \tag{4}$$

where $\zeta$ is the multiplicity label needed in the reduction $(v_1, v_2) \downarrow (T, \mathcal{N})$ of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$. Thus, the vectors (4) should satisfy

$$T_+ \left|\begin{matrix} (v_1, v_2) \\ \zeta\ T = M_T, \mathcal{N} \end{matrix}\right\rangle = 0. \tag{5}$$

Once the basis vector (4) for the highest weight state of $O_T(3)$ with $M_T = T$ is known, the basis vector of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ for any $M_T$ can be expressed in the standard way as

$$\left|\begin{matrix} (v_1, v_2) \\ \zeta\ T, M_T, \mathcal{N} \end{matrix}\right\rangle = \sqrt{\frac{(T+M_T)!}{(2T)!\,(T-M_T)!}}\ (T_-)^{T-M_T} \left|\begin{matrix} (v_1, v_2) \\ \zeta\ T, M_T = T, \mathcal{N} \end{matrix}\right\rangle. \tag{6}$$

In order to find all basis vectors of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ with fixed $M_T > 0$ and $\mathcal{N}$ in the irrep $(v_1, v_2)$ of $O(5)$, one suffices to consider possible irreps $(\Lambda, I)$ of $SU_\Lambda(2) \otimes SU_I(2)$ embedded in the canonical chain satisfying the condition (3). According to the restrictions $M_T = m_\Lambda - m_I$, $\mathcal{N} = m_\Lambda + m_I$ and the reduction rules, we find that the following basis vectors are all possible ones within the $O(5)$ irrep $(v_1, v_2)$ with $M_T \geq 0$ for fixed $\mathcal{N}$:

$$\left|\begin{matrix} & (v_1, v_2) & \\ \Lambda, & & I \\ \frac{1}{2}(\mathcal{N}+M_T), & & \frac{1}{2}(\mathcal{N}-M_T) \end{matrix}\right\rangle \tag{7}$$

with the restrictions

$$\tfrac{1}{2}|\mathcal{N}+M_T| \leq \Lambda \leq \tfrac{1}{2}(v_1+v_2), \quad \tfrac{1}{2}|\mathcal{N}-M_T| \leq I \leq \tfrac{1}{2}(v_1-v_2). \tag{8}$$

Hence, the basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ may be expanded in terms of vectors (7) as

$$\left|\begin{matrix} (v_1, v_2) \\ \zeta\ T = M_T, \mathcal{N} \end{matrix}\right\rangle = \sum_{q=0}^{v_1-v_2} \sum_{p=\mathrm{Max}[0,\ q-v_1+v_2+|\mathcal{N}-T|]}^{\mathrm{Min}[v_1+v_2-q-|\mathcal{N}+T|,\ 2v2]} c_{p,q}^{(\zeta)}$$

$$\times \left|\begin{matrix} & (v_1, v_2) & \\ \Lambda = \frac{1}{2}(v_1+v_2-p-q), & & I = \frac{1}{2}(v_1-v_2+p-q) \\ \frac{1}{2}(\mathcal{N}+T), & & \frac{1}{2}(\mathcal{N}-T) \end{matrix}\right\rangle, \tag{9}$$

where the summations should also be restricted by the condition that $v_1 + v_2 - p - q - |\mathcal{N} + T|$ are even numbers, $\zeta$ is the multiplicity label needed in the reduction $(v_1, v_2) \downarrow (\mathcal{N}, T)$, and $\{c_{pq}^{(\zeta)} \equiv c_{pq}^{(\zeta)}((v_1, v_2), \mathcal{N}, T)\}$ are the expansion coefficients, which must satisfy

$$-\sqrt{\frac{1}{2}} T_+ \left|\begin{matrix} (v_1, v_2) \\ \zeta\ T = M_T, \mathcal{N} \end{matrix}\right\rangle = U_{\frac{1}{2}-\frac{1}{2}} \left|\begin{matrix} (v_1, v_2) \\ \zeta\ T = M_T, \mathcal{N} \end{matrix}\right\rangle = 0. \tag{10}$$

By using the explicit matrix elements of $\mathbf{U}$ in the $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ basis provided in Refs. [6–9], Eq. (10) leads to the following four-term relation to determine the expansion coefficients $\{c^{(\zeta)}_{p,q}\}$:

$$\begin{aligned}
&c^{(\zeta)}_{p,q+1}(-1)^{2\mathcal{N}-2q+2v_1}\\
&\quad\times\left[\frac{(1+q)(2v_1-q+2)(v_1+v_2-q+1)(v_1+v_2-p-q+T+\mathcal{N}+1)(v_1-v_2+T-\mathcal{N}+p-q+1)(v_1-v_2-q)}{(v_1+v_2-p-q)(v_1-v_2+p-q)}\right]^{\frac{1}{2}}\\
&+c^{(\zeta)}_{p+1,q}(-1)^{v_1+v_2+\mathcal{N}-p-q+T}\\
&\quad\times\left[\frac{(1+p)(2v_2-p)(v_1+v_2-p+1)(v_1+v_2+T+\mathcal{N}-p-q+1)(v_1-v_2+p+2)(v_1-v_2-T+\mathcal{N}+p-q+1)}{(v_1+v_2-p-q)(v_1-v_2+p-q+2)}\right]^{\frac{1}{2}}\\
&+c^{(\zeta)}_{p-1,q}(-1)^{v_1-v_2+\mathcal{N}+p-q-T}\\
&\quad\times\left[\frac{p(2v_2-p+1)(v_1+v_2-p+2)(v_1+v_2-T-\mathcal{N}-p-q+1)(v_1-v_2+p+1)(v_1-v_2+T-\mathcal{N}+p-q+1)}{(v_1+v_2-p-q+2)(v_1-v_2+p-q)}\right]^{\frac{1}{2}}\\
&+c^{(\zeta)}_{p,q-1}\\
&\quad\times\left[\frac{q(2v_1-q+3)(v_1+v_2-q+2)(v_1+v_2-T-\mathcal{N}-p-q+1)(v_1-v_2-T+\mathcal{N}+p-q+1)(v_1-v_2-q+1)}{(v_1+v_2-p-q+2)(v_1-v_2+p-q+2)}\right]^{\frac{1}{2}}=0.
\end{aligned} \tag{11}$$

Accordingly, one can construct a matrix equation equivalent to Eq. (11),

$$\mathbf{P}\big((v_1,v_2),\mathcal{N},T\big)\,\mathbf{c}^{(\zeta)}=\Lambda\mathbf{c}^{(\zeta)}. \tag{12}$$

Entries of the isospin projection matrix $\mathbf{P}\big((v_1,v_2),\mathcal{N},T\big)$ can easily be read out from Eq. (11) and the eigenvector $\mathbf{c}^{(\zeta)} \equiv \mathbf{c}^{(\zeta)}\big((v_1,v_2),\mathcal{N},T\big)$, which transpose is arranged as $\big(\mathbf{c}^{(\zeta)}\big)^{\mathrm{T}} = (c^{(\zeta)}_{0,0}, c^{(\zeta)}_{1,0}, c^{(\zeta)}_{2,0}, \dots, c^{(\zeta)}_{0,1}, c^{(\zeta)}_{1,1}, \dots)$. The components of the eigenvector $\mathbf{c}^{(\zeta)}$ corresponding to $\Lambda = 0$ provide the expansion coefficients $\{c^{(\zeta)}_{p,q}\}$ of Eq. (9). Once the matrix $\mathbf{P}\big((v_1,v_2),\mathcal{N},T\big)$ is constructed, it can be verified that the number of $\Lambda = 0$ solutions of Eq. (12) equals exactly to the number of rows of $\mathbf{P}\big((v_1,v_2),\mathcal{N},T\big)$ with all entries zero. Actually, the eigenvectors $\mathbf{c}^{(\zeta)}\big((v_1,v_2),T,\mathcal{N}\big)$ belong to the null-space of $\mathbf{P}\big((v_1,v_2),\mathcal{N},T\big)$. Since there are many ways to find null-space vectors of a matrix, to find solutions of Eq. (12) with $\Lambda = 0$ becomes practically easy. Furthermore, $\big(\mathbf{c}^{(\zeta')}\big)^{\mathrm{T}}\cdot\mathbf{c}^{(\zeta)} \neq 0$ when the multiplicity is greater than 1 mainly because the projection matrix $\mathbf{P}\big((v_1,v_2),\mathcal{N},T\big)$ is nonsymmetric. Therefore, the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis vectors (9) constructed from the expansion coefficients obtained according to Eq. (11) are also non-orthogonal with respect to the multiplicity label $\zeta$ in general. The Gram–Schmidt process may be adopted in order to construct orthonormalized basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$. Nevertheless, in the Wolfram Mathematica, the built-in function NullSpace of a matrix with non-integer entries generates orthonormalized null-space vectors automatically, with which the Gram–Schmidt orthogonalization can be avoid. In the following, we use $\tilde{\mathbf{c}}^{(\zeta)}$ to denote the orthonormalized null-space vectors of $N\big[\mathbf{P}\big((v_1,v_2),\mathcal{N},T\big)\big]$ with respect to the multiplicity label $\zeta$ obtained from the Wolfram Mathematica numerically, where $N[\mathbf{P}]$ means to take $\mathbf{P}$ with numerical valued entries with a default precision.

The CPU time cost and memory space needed for a computer to solve the null-space problem (12) depend mainly on the number of terms $d(\mathcal{N},T)$ needed in the expansion (9), which equals to the number of columns of $\mathbf{P}\big((v_1,v_2),\mathcal{N},T\big)$. Generally, it would take the CPU time on the order of $\mathrm{O}(d^3)$ with a unit inversely proportional to the CPU frequency, and the memory space on the order of $\mathrm{O}(d^2)$ bytes. When $v_1$ and $v_2$ are integers, for example, we observe form Eq. (9) that the maximal number of terms occurs in the $T = \mathcal{N} = 0$ case. In such extreme case, the upper bound of the number of terms involved in the expansion can be estimated as

$$d(\mathcal{N}=0, T=0) \leq \sum_{q=0}^{v_1-v_2} \sum_{p=\mathrm{Max}[0,\; q-v_1+v_2]}^{\mathrm{Min}[v_1+v_2-q,\; 2v_2]} 1 = (1+v_1-v_2)(2v_2+1), \tag{13}$$

which shows that $\mathrm{Max}[d(\mathcal{N}, T)] \leq d(\mathcal{N}=0, T=0)$ increases with $v_1$ linearly and with $v_2$ quadratically.

## 3 Matrix elements of the isovector pairing operators in the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis

Once the orthonormalized expansion coefficients $\{\tilde{\mathbf{c}}^{(\zeta)}\}$ are obtained according to the isospin projection shown in the previous section, one can easily calculate matrix elements of $O(5)$ generators $\{A^\dagger_\mu,\ A_\mu,\ T_\mu, \mathcal{N}\}$ $(\mu = -1,\ 0,\ 1)$ given in Eq. (1) in the $O_T(3) \otimes O_{\mathcal{N}}(2)$ basis. Since the matrix elements of $\{T_\mu,\ \mathcal{N}\}$ are well-known, depend only on $T$ or $\mathcal{N}$, and are irrelevant to the irrep of $O(5)$ and the multiplicity label $\zeta$, only the formulae of matrix elements of the isovector pairing operator $A^\dagger_\mu$ and $A_\mu$ in the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis will be provided.

In the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis, the pair creation operators $\mathcal{A}^+_\mu$ with $\{\mathcal{A}^+_1 = -A^\dagger_1,\ \mathcal{A}^+_0 = A^\dagger_0,\ \mathcal{A}^+_{-1} = A^\dagger_{-1}\}$ and the pair annihilation operators $\mathcal{A}_\mu$ with $\{\mathcal{A}_1 = A_{-1},\ \mathcal{A}_0 = -A_0,\ \mathcal{A}_{-1} = -A_1\}$ are $T = 1$ irreducible tensor operators of $O_T(3)$ satisfying the following conjugation relation [18]:

$$\mathcal{A}_\mu = (-1)^{1-\mu} \big(\mathcal{A}^+_{-\mu}\big)^\dagger. \tag{14}$$

These $T = 1$ irreducible tensor operators shift $\mathcal{N}$ by one unit, while $A^\dagger_1 = \nu_+$, $A^\dagger_0 = U_{\frac{1}{2}\frac{1}{2}}$, and $A^\dagger_{-1} = \tau_+$ in the $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ basis shown in Refs. [6–9]. Using the Wigner–Eckart theorem for matrix elements of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$, we have

$$\left\langle \begin{matrix} (v_1, v_2) \\ \zeta'\, T'\, M'_T, \mathcal{N}' \end{matrix} \middle| \mathcal{A}^+_\mu \middle| \begin{matrix} (v_1, v_2) \\ \zeta\, T\, M_T, \mathcal{N} \end{matrix} \right\rangle = \delta_{\mathcal{N}', \mathcal{N}+1} \langle T M_T, 1\mu | T'\ M'_T \rangle \left\langle \begin{matrix} (v_1, v_2) \\ \zeta'\, T', \mathcal{N}+1 \end{matrix} \middle\| \mathcal{A}^+ \middle\| \begin{matrix} (v_1, v_2) \\ \zeta\, T, \mathcal{N} \end{matrix} \right\rangle, \tag{15}$$

where $\langle T M_T, 1\mu | T'\ M'_T \rangle$ is the Clebsch–Gordan coefficient of $O_T(3)$, and

$$\left\langle \begin{matrix} (v_1, v_2) \\ \zeta'\, T', \mathcal{N}' \end{matrix} \middle\| \mathcal{A}^+ \middle\| \begin{matrix} (v_1, v_2) \\ \zeta\, T, \mathcal{N} \end{matrix} \right\rangle$$

is the $O_T(3)$-reduced matrix element. In the calculation, we ensure that $T'$ is always involved in the $O_T(3)$ coupling $T \otimes 1$, and $M'_T = M_T + \mu$ is always satisfied. By using Eq. (9) and the expressions of $A^\dagger_\mu$ in terms of the generators of $O(5)$ in the $SU_\Lambda(2) \otimes SU_I(2)$ basis shown in Eq. (1), the left-hand-side of Eq. (15) can be expressed in terms of expansion coefficients $\tilde{\mathbf{c}}^{(\zeta)}$ and the matrix elements of $O(5)$ generators in the $SU_\Lambda(2)\otimes SU_I(2)$ basis. In the following, we list nonzero $O_T(3)$-reduced matrix elements of $\mathcal{A}^\dagger$ derived in this way:

$$\left\langle \begin{matrix}(v_1,v_2)\\ \zeta'\,T+1,\mathcal{N}+1\end{matrix} \middle\| \mathcal{A}^+ \middle\| \begin{matrix}(v_1,v_2)\\ \zeta\,T,\mathcal{N}\end{matrix}\right\rangle = -\frac{1}{2}\sum_{q,p} \tilde{c}^{(\zeta')}_{p,q}(\mathcal{N}+1,T+1)\,\tilde{c}^{(\zeta)}_{p,q}(\mathcal{N},T)$$

$$\times\sqrt{(v_1+v_2-p-q-\mathcal{N}-T)(v_1+v_2-p-q+\mathcal{N}+T+2)},$$

$$\left\langle \begin{matrix}(v_1,v_2)\\ \zeta'\,T,\mathcal{N}+1\end{matrix} \middle\| \mathcal{A}^+ \middle\| \begin{matrix}(v_1,v_2)\\ \zeta\,T,\mathcal{N}\end{matrix}\right\rangle$$
$$= \sqrt{\tfrac{T+1}{8T}}\sum_{q,p}\tilde{c}^{(\zeta)}_{q,t}(\mathcal{N},T)\Big(\tilde{c}^{(\zeta')}_{p,q-1}(\mathcal{N}+1,T)\,(-1)^{2\mathcal{N}-2q+2v_1+1}$$
$$\times\left[\tfrac{q(2v_1-q+3)(v_1-v_2+p-q+\mathcal{N}-T+2)(v_1+v_2-p-q+T+\mathcal{N}+2)(v_1-v_2-q+1)(v_1+v_2-q+2)}{(v_1-v_2+p-q+1)(v_1-v_2+p-q+2)(v_1+v_2-p-q+1)(v_1+v_2-p-q+2)}\right]^{\frac{1}{2}}$$
$$+\,\tilde{c}^{(\zeta')}_{p-1,q}(\mathcal{N}+1,T)(-1)^{v_1+v_2+\mathcal{N}-p-q+T}$$
$$\times\left[\tfrac{p(2v_2-p+2)(v_1-v_2+p-q-\mathcal{N}+T)(v_1+v_2-p-q+T+\mathcal{N}+2)(v_1-v_2+p+1)(v_1+v_2-p+2)}{(v_1-v_2+p-q+1)(v_1-v_2+p-q)(v_1+v_2-p-q+1)(v_1+v_2-p-q+2)}\right]^{\frac{1}{2}}$$
$$+\,\tilde{c}^{(\zeta')}_{p+1,q}(\mathcal{N}+1,T)(-1)^{v_1-v_2+\mathcal{N}+p-q-T}$$
$$\times\left[\tfrac{(p+1)(2v_2-p)(v_1-v_2+\mathcal{N}-T+p-q+2)(v_1+v_2-T-\mathcal{N}-p-q)(v_1-v_2+p+2)(v_1+v_2-p+1)}{(v_1-v_2+p-q+1)(v_1-v_2+p-q+2)(v_1+v_2-p-q)(v_1+v_2-p-q+1)}\right]^{\frac{1}{2}}$$
$$+\,\tilde{c}^{(\zeta')}_{p,q+1}(\mathcal{N}+1,T)$$
$$\times\left[\tfrac{(q+1)(2v_1-q+2)(v_1-v_2-T+\mathcal{N}+p-q)(v_1+v_2-T-\mathcal{N}-p-q)(v_1-v_2-q)(v_1+v_2-q+1)}{(v_1-v_2+p-q)(v_1-v_2+p-q+1)(v_1+v_2-p-q)(v_1+v_2-p-q+1)}\right]^{\frac{1}{2}}\Big)$$

for $T\geq\frac{1}{2}$, and

$$\left\langle \begin{matrix}(v_1,v_2)\\ \zeta'\,T-1,\mathcal{N}+1\end{matrix} \middle\| \mathcal{A}^+ \middle\| \begin{matrix}(v_1,v_2)\\ \zeta\;T,\mathcal{N}\end{matrix}\right\rangle$$

$$=\frac{1}{2}\sqrt{\frac{2T+1}{2T-1}}\sum_{q,p}\tilde{c}^{(\zeta')}_{p,q}(\mathcal{N}+1,T-1)\tilde{c}^{(\zeta)}_{p,q}(\mathcal{N},T)$$

$$\times\sqrt{(v_1-v_2+p-q-\mathcal{N}+T)(v_1-v_2+p-q+\mathcal{N}-T+2)} \quad (16)$$

for $T\geq 1$. The other non-zero reduced matrix elements of $\mathcal{A}$ can be obtained by the conjugation relation:

$$\left\langle \begin{matrix}(v_1,v_2)\\ \zeta'\,T',\mathcal{N}'\end{matrix} \middle\| \mathcal{A} \middle\| \begin{matrix}(v_1,v_2)\\ \zeta\,T,\mathcal{N}\end{matrix}\right\rangle = (-1)^{T'-T+1}\sqrt{\frac{2T+1}{2T'+1}}\left\langle \begin{matrix}(v_1,v_2)\\ \zeta\,T,\mathcal{N}\end{matrix} \middle\| \mathcal{A}^+ \middle\| \begin{matrix}(v_1,v_2)\\ \zeta'\,T',\mathcal{N}'\end{matrix}\right\rangle. \quad (17)$$

# 4 Applications to the isovector pairing model

In the spherical shell model, we consider $n$ valence nucleons with $J=0$ and $T=1$ pairing interactions in $p$ single-particle orbits. In general, the spherical shell model is the mean-field plus the isovector pairing interaction Hamiltonian may be written as [5]

$$\hat{H}=\sum_j \epsilon_j n_j - G_\pi A^\dagger_{+1}A_{+1} - G_{\pi\nu}A^\dagger_0 A_0 - G_\nu A^\dagger_{-1}A_{-1}, \quad (18)$$

where $\epsilon_j$ is the single particle energy of the $j$-orbit, $G_\pi > 0$, $G_\nu > 0$, and $G_{\pi\nu} > 0$ are proton-proton ($pp$), neutron-neutron ($nn$), and neutron-proton ($np$) pairing interaction strengths, respectively, $n_j = \sum_{mm_t} a^\dagger_{jm,m_t} a_{jm,m_t}$ is the valence nucleon number operator in the $j$-orbit, and $a^\dagger_{jm,m_t}$ ($a_{jm,m_t}$) is the creation (annihilation) operator for a valence nucleon in the state with angular momentum $j$, angular momentum projection $m$, and isospin projection $m_t = 1/2,\ -1/2$. When $G_\pi = G_\nu = G_{\pi\nu} = G$, the isospin is a good quantum number. In this isospin conserving-case, the Hamiltonian (18) is exactly solvable [18,19]. Since neutron and proton single-particle energies in the $j$-orbit are the same, it is expected that $G_\pi = G_\nu = G$ may be approximately satisfied, while, in general, $G_{\pi\nu} \neq G$ and the Bethe ansatz method used in Ref. [18,19] will no longer be useful. In such a case, the Hamiltonian (18) may be diagonalized in the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis [20–23]. For the sake of simplicity, in the following, we consider the degenerate case with $\epsilon_j = \epsilon\ \forall\ j$ when the first term in Eq. (18) becomes a constant for a fixed number of nucleons $n$, and is neglected. Thus, the Hamiltonian can be expressed as

$$\hat{H}_P = -G\,\mathcal{A}^+ \cdot \mathcal{A}, \tag{19}$$

where $G_\pi = G_\nu = G_{\pi\nu} = G$ is assumed. The Hamiltonian (19) is $O_T(3)$ invariant and can be expressed as

$$\hat{H}_{O_T(3)} = \hat{H}_P = -G\,\mathcal{A}^+ \cdot \mathcal{A} = -\frac{1}{2} G \left( C_2\big(O(5)\big) - \hat{\mathcal{N}}(\hat{\mathcal{N}} - 3) - \mathbf{T} \cdot \mathbf{T} \right), \tag{20}$$

which is diagonal in the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis:

$$\hat{H}_{O_T(3)} \left| \begin{matrix} (v_1, v_2) \\ \zeta\, T, M_T, \mathcal{N} \end{matrix} \right\rangle = -\frac{1}{2} G \big( v_1(v_1+3) + v_2(v_2+1) - \mathcal{N}(\mathcal{N}-3) - T(T+1) \big) \left| \begin{matrix} (v_1, v_2) \\ \zeta\, T, M_T, \mathcal{N} \end{matrix} \right\rangle. \tag{21}$$

In this case, the labels of the $O(5)$ irrep $(v_1, v_2)$ are related to the seniority of nucleons $v$ and the reduced isospin $t$ with $v_1 = \Omega - v/2$ and $v_2 = t$, where $v$ and $t$ indicate that there are $v$ nucleons coupled to the isospin $t$, which are not included in the $J = 0$ and $T = 1$ pairs. One can also directly calculate matrix elements of $\mathcal{A}^+ \cdot \mathcal{A}$ in the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis using the matrix elements of $\mathcal{A}^+$ provided in the previous Section,

$$\left\langle \begin{matrix} (v_1, v_2) \\ \zeta\, T, M_T, \mathcal{N} \end{matrix} \right| \mathcal{A}^+ \cdot \mathcal{A} \left| \begin{matrix} (v_1, v_2) \\ \zeta\, T, M_T, \mathcal{N} \end{matrix} \right\rangle = \sum_{\zeta' T'} \left| \left\langle \begin{matrix} (v_1, v_2) \\ \zeta\, T, \mathcal{N} \end{matrix} \right\| \mathcal{A}^+ \left\| \begin{matrix} (v_1, v_2) \\ \zeta'\, T', \mathcal{N} - 1 \end{matrix} \right\rangle \right|^2, \tag{22}$$

where the relation (17) is used to check that the results shown in the previous Section are indeed valid.

Moreover, besides the $O_T(3)$ isospin dynamical symmetry limit case shown above, there is the well known $SU_\Lambda(2) \otimes SU_I(2)$ quasispin dynamical symmetry limit for any value of $G_\pi$ and $G_\nu$ when $G_{\pi\nu} = 0$, where $\Lambda$ and $I$ are the quasi-spin of the proton and neutron pairing, respectively. In this case, the pairing interaction part of Eq. (18)

$$\hat{H}_{SU_\Lambda(2) \otimes SU_I(2)} = -G_\pi A^\dagger_{+1} A_{+1} - G_\nu A^\dagger_{-1} A_{-1} \tag{23}$$

is diagonal in the $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ basis,

$$\hat{H}_{SU_\Lambda(2)\otimes SU_I(2)} \left| \begin{matrix} (v_1, v_2) \\ \Lambda, I \\ m_\Lambda, m_I \end{matrix} \right\rangle$$

$$= \Big(-G_\pi\big(\Lambda(\Lambda+1) - m_\Lambda(m_\Lambda+1)\big) - G_\nu\big(I(I+1) - m_I(m_I+1)\big)\Big) \left| \begin{matrix} (v_1, v_2) \\ \Lambda, I \\ m_\Lambda, m_I \end{matrix} \right\rangle, \quad (24)$$

where $\Lambda = (\Omega - v_\pi)/2$ and $I = (\Omega - v_\nu)/2$ and $v_\pi$ ($v_\nu$) is the proton (neutron) seniority, $m_\Lambda = n_\pi/2 - \Omega/2$, $m_I = n_\nu/2 - \Omega/2$ and $n_\pi$ ($n_\nu$) is the number of valence protons (neutrons), which shows that the Hamiltobian (23) is still block diagonal with respect to the O(5) irrep labeled by $(v_1, v_2)$, though the interpretation of $(v_1, v_2)$ in terms of $v$ and $t$ is no longer appropriate in this case due to the fact that the isospin symmetry is broken.

For other values of the pairing interaction strengths, the pairing interaction part of the Hamiltobian (18) can be only diagonalized in any basis of $O(5)$ and the eigenstates may be expanded in terms of either the basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ or those of $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$. The parameter rectangle of the pure isovector pairing Hamiltonian is illustrated in Fig. 1, which shows that the pure isovector pairing Hamiltonian may be diagonalized in the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis, except the $G_{\pi\nu} = 0$ case indicated by the left leg of the rectangle with the $SU_\Lambda(2) \otimes SU_I(2)$ quasispin dynamical symmetry.

## 5 Summary

In this talk, a simple and effective algebraic isospin projection procedure for constructing basis vectors of the irreducible representations of the non-canonical

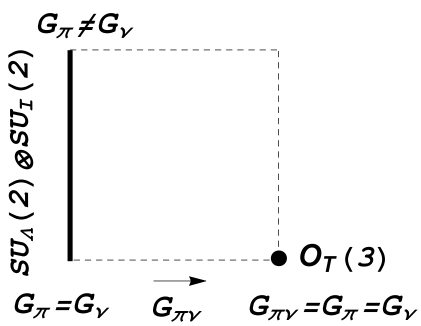


Figure 1: The parameter rectangle of the isovector pairing Hamiltonian, where the left leg marked by the solid line represents the Hamiltonian with arbitrary values of $G_\pi$ and $G_\nu$ and $G_{\pi\nu} = 0$ corresponding to the $SU_\Lambda(2) \otimes SU_I(2)$ quasispin dynamical symmetry, and the vertex marked by the solid dot represents the Hamiltonian with $G_\pi = G_\nu = G_{\pi\nu}$ corresponding to the $O_T(3)$ isospin dynamical symmetry. The Hamiltonian for other values of the parameters shown by the other area of the rectangle may be diagonalized in either the $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ or the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis.

$O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis from those of the canonical $O(5) \supset SU_\Lambda(2) \otimes SU_I(2)$ basis is presented. The main content of this talk is based on our recent work [24], where more detailed results are provided. It is shown that the expansion coefficients can be obtained as components of the null-space vectors of the projection matrix, where there are only four nonzero elements in each row in general. There are currently available well-optimized algorithms for computing the null-space vectors of a matrix, for example, the Wolfram Mathematica providing the null-space vectors which are orthonormalized. Hence, an evaluation of the expansion coefficients of the orthonormal basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ in terms of the basis of the canonical chain becomes straightforward. The advantage of this work lies in the fact that the basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ thus obtained are orthonormalized with respect to the $O(5) \downarrow O_T(3) \otimes O_{\mathcal{N}}(2)$ branching multiplicity label $\zeta$ for any irrep of $O(5)$. Explicit formulae for evaluating $O_T(3)$-reduced matrix elements of $O(5)$ generators are derived.

For the general non-degenerate case of the Hamiltonian (18) when there are $p$ non-degenerate orbits, one needs to diagonalize the Hamiltonian in the $\bigotimes_{i=1}^{p} O_i(5)$ subspace, where the matrix elements of the isovector pairing operators provided in this talk are useful. Thus, one can further analyze the isospin symmetry breaking effects in the Hamiltonian (18) with $G_\pi \neq G_\nu \neq G_{\pi\nu}$ as was done for the specific cases in Refs. [20–23], which is also helpful for understanding the $np$-pairing effect in $N \sim Z$ nuclei [25].

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## References

[1] K. Helmers, Nucl. Phys. **23**, 594 (1961).

[2] A. K. Kerman, Ann. Phys. **12**, 300 (1961).

[3] M. Ichimura, Prog. Theor. Phys. **33**, 215 (1965).

[4] J. N. Ginocchio, Nucl. Phys. **74**, 321 (1965).

[5] K. T. Hecht, Phys. Rev. **139**, B794 (1965).

[6] K. T. Hecht, Nucl. Phys. **63**, 214 (1965).

[7] R. T. Sharp and S. C. Pieper, J. Math. Phys. **9**, 663 (1968).

[8] N. Kemmer, D. L. Pursey and S. A. Williams, J. Math. Phys. **9**, 1224 (1968).

[9] F. Pan, Chin. Phys. **11**, 870 (1991).

[10] R. P. Hemenger and K. T. Hecht, Nucl. Phys. A **145**, 468 (1970).

[11] K. Ahmed and R. T. Sharp, J. Math. Phys. **11**, 1112 (1970).

[12] Yu F. Smirnov and V. N. Tolstoy, Rep. Math. Phys. 4, 97 (1973).

[13] S. Ališauskas, J. Phys. A **17**, 2899 (1984).

[14] K. T. Hecht and J. P. Elliott, Nucl. Phys. A **438**, 29 (1985).

[15] Q.-Z. Han, H.-Z. Sun and J.-J. Wang, Commun. Theor. Phys. **20**, 201 (1993).

[16] M. A. Caprio, K. D. Sviratcheva and A. E. McCoy, J. Math. Phys. **51**, 093518 (2010).

[17] M. A. Caprio, J. Phys. Conf. Ser. **237**, 012009 (2010).

[18] F. Pan and J. P. Draayer, Phys. Rev. C **66**, 044314 (2002).

[19] J. Dukelsky, V. G. Gueorguiev, P. Van Isacker, S. Dimitrova, B. Errea and S. Lerma H., Phys. Rev. Lett. **96**, 072503 (2006).

[20] K. D. Sviratcheva, A. I. Georgieva and J. P. Draayer, J. Phys. G **29**, 1281 (2003).

[21] K. D. Sviratcheva, A. I. Georgieva and J. P. Draayer, Phys. Rev. C **69**, 024313 (2004).

[22] K. D. Sviratcheva, A. I. Georgieva and J. P. Draayer, Phys. Rev. C **70**, 064302 (2004).

[23] K. D. Sviratcheva, J. P. Draayer and J. P. Vary, Phys. Rev. C **73**, 034324 (2006).

[24] F. Pan, X. Ding, K. D. Launey and J. P. Draayer, Nucl. Phys. A **974**, 86 (2018).

[25] S. Frauendorf and A. O. Macchiavelli, Prog. Part. Nucl. Phys. **78**, 24 (2014).