Unphysical Sheets and Resonances for the Friedrichs–Faddeev Model

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Abstract

The Friedrichs–Faddeev model is considered in the case where the kernel of the potential operator is holomorphic in both arguments on a certain complex domain. For this model, we, first, derive representations for the transition operator and scattering matrix continued on unphysical energy sheet(s) that explicitly express them in terms of the same operators exclusively on the physical sheet. Then the Friedrichs–Faddeev Hamiltonian becomes subject to a complex deformation. We show that, in the case under consideration, the deformation resonances (non-real eigenvalues of the deformed Hamiltonian) are nothing but the scattering matrix resonances, i. e., they represent the poles of the scattering matrix analytically continued on the respective unphysical energy sheet.

Keywords: *Friedrichs–Faddeev model; complex deformation; resonances; unphysical sheets*

1 Introduction

Assume that $\mathfrak{h}$ is a Hilbert space and let $\Delta = (a, b)$, where $-\infty \le a < b \le \infty$. Denote by $L_2(\Delta, \mathfrak{h})$ the Hilbert space of $\mathfrak{h}$-valued functions of $\lambda \in (a, b)$ with the scalar product

$$\langle f, g\rangle = \int_a^b d\lambda\, \langle f(\lambda), g(\lambda)\rangle_{\mathfrak{h}},$$

where $\langle\cdot,\cdot\rangle_{\mathfrak{h}}$ stands for the scalar product in $\mathfrak{h}$. The Hamiltonian of the Friedrichs–Faddeev model has the form

$$H = H_0 + V \tag{1.1}$$

with H_0 being the operator of multiplication by an independent variable in $L_2(\Delta, \mathfrak{h})$,

$$(H_0 f)(\lambda) = \lambda f(\lambda), \qquad \lambda \in \Delta, \qquad f \in L_2(\Delta, \mathfrak{h}), \tag{1.2}$$

and V being an integral operator,

$$(Vf)(\lambda) = \int_a^b V(\lambda, \mu)\, f(\mu)\, d\mu. \tag{1.3}$$

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http://www.ntse.khb.ru/files/uploads/2018/proceedings/Motovilov.pdf.

It is assumed that, for every $\lambda, \mu \in \Delta$, the quantity $V(\lambda, \mu)$ is a bounded linear operator on $\mathfrak{h}$ such that $V(\lambda, \mu) = V(\mu, \lambda)^*$, and, in addition, V is a Hölder continuous operator-valued function of $\lambda, \mu \in \overline{\Delta}$. Furthermore, one requires

$$V(a, \mu) = V(b, \mu) = V(\lambda, a) = V(\lambda, b) = 0 \qquad \text{in the case of finite } a \text{ or/and } b \quad (1.4)$$

or imposes suitable requirements on the rate of decreasing of $V(\lambda, \mu)$ as $|\lambda|, |\mu| \to \infty$ in the case of infinite a or/and b.

In its starting form the model (1.1)–(1.4) was introduced by K. Friedrichs [1] who considered the Hamiltonian

$$H_\epsilon = H_0 + \epsilon V, \qquad \epsilon > 0, \quad (1.5)$$

with H_0 and V given by (1.2) and (1.3) in the simplest case of one-dimensional internal Hilbert space $\mathfrak{h} = \mathbb{C}$ and $\Delta = (-1, 1)$. The spectrum of the (self-adjoint) operator H_0 is absolutely continuous and, in this case, coincides with the segment $[-1, 1]$. Friedrichs studied variation of the continuous spectrum of H_0 under the perturbation ϵV. He has succeeded to prove that, if ϵ is sufficiently small, then the spectrum of H_ϵ remains absolutely continuous and still fills the segment $[-1, 1]$. In Ref. [2], Friedrichs has extended this result to the case of arbitrary finite- or infinite-dimensional Hilbert space $\mathfrak{h}$ and arbitrary finite or infinite end points a and b. More precisely, he has proven that, if $\epsilon > 0$ is small enough, then the perturbed operator (1.5) is unitarily equivalent to the unperturbed one, H_0, and, hence, the spectrum of H_ϵ is absolutely continuous and fills the set $\overline{\Delta}$.

O. A. Ladyzhenskaya and L. D. Faddeev have dropped in Ref. [3] the assumption of smallness of the perturbation V and studied the model Hamiltonian (1.1)–(1.4) with not small ϵ at V. However, instead of the smallness, they required compactness of the values of $V(\lambda, \mu)$ as operators on $\mathfrak{h}$ for all $\lambda, \mu \in \Delta$. Detail proofs for the results of Ref. [3] are presented by Faddeev in Ref. [4]. As a matter of fact, the paper [4] contains a complete version of the scattering theory for the model (1.1)–(1.4). Furthermore, the paper [4] may be viewed as a relatively simple introduction to the approach used by Faddeev in his celebrated study [5] of the three-body problem. Also notice that the typical two-body Schrödingrer operator may be reduced to the Friedrichs–Faddeev model with $a = 0$, $b = +\infty$ and $\mathfrak{h} = L_2(S^2)$, where S^2 is the unit sphere in $\mathbb{R}^3$ (see Ref. [4]; cf. Ref. [6, Section 3]).

Faddeev's in-depth study [4] of the Hamiltonian (1.1)–(1.4) is the main reason why this Hamiltonian is often referred to as the Friedrichs–Faddeev model. In addition, the double naming allows to distinguish the model (1.1)–(1.4) from another popular model due to Friedrichs contained in Ref. [2]. The second model from Ref. [2] involves a 2×2 block matrix Hamiltonian and works well, in particular, in the theory of Feshbach resonances (see, e.g., Refs. [7, 8] and references therein). For later results just on the Friedrichs–Faddeev model and its generalizations, see Refs. [9–13].

In the present work, we adopt the ideas and approach from the previous works of the author [14, 15] in order to study the structure of the T- and S-matrices for the Friedrichs–Faddeev model continued on unphysical energy sheets neighboring the physical one. Namely, we obtain representations that explicitly express the continued T- and S-matrices in terms of the same operators considered exclusively on the physical sheet (see Lemmas 2.2 and 2.3 below). The obtained representations show, in particular, that a resonance on an unphysical sheet under consideration corresponds to the energy z in the physical sheet where the scattering matrix has the zero eigenvalue.

We perform a complex deformation of the Friedrichs–Faddeev Hamiltonian. (Notice that the "usual" complex scaling [16, 17] may be understood as a particular case of the complex deformation, see Ref. [6, Section 3].) A complex discrete spectrum of the complexly-deformed Hamiltonian is interpreted as resonances. We show that these resonances are simultaneously the poles of the continued scattering (and T-) matrix on the unphysical sheet(s), that is, they are resonances in the sense of scattering theory. Recall that, in general, to prove the equivalence of the scaling resonances and scattering matrix resonances is a rather hard job (see Ref. [18]). In contrast, in the case of the Friedrichs–Faddeev model, the proof of such an equivalence is quite easy and illustrative.

Throughout the article, we denote by $\sigma(T)$ the spectrum of a closed linear operator T. Notation T^* is used for the adjoint of T. T is called self-adjoint (Hermitian) if $T^* = T$. Notations $\sigma_p(T)$ and $\sigma_c(T)$ are used for the point and continuous spectra of T, respectively. By $I_{\mathfrak{K}}$ we denote the identity operator on a vector space $\mathfrak{K}$; the index $\mathfrak{K}$ is omitted if no confusion arises. Notation $\mathbb{C}^+ = \{z \in \mathbb{C} \,|\, \mathrm{Im} z > 0\}$ ($\mathbb{C}^- = \{z \in \mathbb{C} \,|\, \mathrm{Im} z < 0\}$) is used for the upper (lower) halfplane of of the complex plane $\mathbb{C}$.

The present paper represents a conference version of the work [6].

2 Structure of the T- and S-matrices on unphysical energy sheets

We consider the model (1.1)–(1.4) in the case where for each $\lambda, \mu \in (a, b)$ the value of $V(\lambda, \mu)$ is a compact operator in $\mathfrak{h}$. We assume, in addition, that the function $V(\lambda, \mu)$ admits analytic continuation both in λ and μ onto a domain $\Omega \subset \mathbb{C}$ containing Δ. More precisely, we suppose that

$$V(\lambda, \mu) \quad \text{is compact and holomorphic in both} \quad \lambda, \mu \in \Omega, \quad \Omega \supset (a, b). \tag{2.1}$$

Also we assume that $V(\lambda, \mu) = V(\mu, \lambda)^*$ for real $\lambda, \mu \in \Delta$ (which is needed for the self-adjointness of V). Surely, this implies $V(\lambda, \mu) = V(\mu^*, \lambda^*)^*$ for any $\lambda, \mu \in \Omega$ such that their conjugates $\lambda^*, \mu^* \in \Omega$ and, hence, the domain Ω should be mirror-symmetric with respect to the real axis.

Following Refs. [2, 4] one imposes some natural requirements on the rate of decreasing of $V(\lambda, \mu)$ as $|\lambda|, |\mu| \to \infty$ in the case of $a = -\infty$ or/and $b = +\infty$. To unify the consideration, we simply assume that

$$\|V(\lambda, \mu)\| \le K(1 + |\lambda| + |\mu|)^{-(1+\eta_1)}, \qquad \eta_1 > 0; \tag{2.2}$$

$$\|V(\lambda + \alpha, \mu + \beta) - V(\lambda, \mu)\| \le K(1 + |\lambda| + |\mu|)^{-(1+\eta_1)}(|\alpha|^{\eta_2} + |\beta|^{\eta_2}), \quad \eta_2 > 1/2, \tag{2.3}$$

with some $K > 0$ for any $\lambda, \mu \in \Omega$ and any α, β such that $\lambda + \alpha \in \Omega$, $\mu + \beta \in \Omega$. Since $V(\lambda, \mu)$ is holomorphic in both $\lambda \in \Omega$ and $\mu \in \Omega$, the requirement (2.3) with $\eta_2 < 1$ is essential only in the neighborhoods of the finite end points a and/or b. Otherwise, one may replace η_2 with unity.

We use the standard notations for the resolvents,

$$R_0(z) := (H_0 - z)^{-1}, \qquad R(z) := (H - z)^{-1},$$

and for the transition operator,

$$T(z) := V - VR(z)V. \tag{2.4}$$

Since, at least for $z \notin \sigma(H_0) \cup \sigma(H)$,

$$R(z) = R_0(z) - R_0(z)T(z)R_0(z), \tag{2.5}$$

the study of the spectral problem for the perturbed Hamiltonian $H = H_0 + V$ is reduced to the study of the transition operator (T-matrix) $T(z)$, the kernel of which is less singular than that of the resolvent $R(z)$.

From [4, Theorem 3.1] it follows that the kernel $T(\lambda, \mu, z)$ is a well-behaved function of $\lambda, \mu \in \Delta$ and z on the complex plane $\mathbb{C}$ punctured at $\sigma_p(H)$ and cut along $[a, b]$. Moreover, $T(\lambda, \mu, z)$ is of the same class (2.2), (2.3) as $V(\lambda, \mu)$ but with η_1 and η_2 replaced by positive $\eta_1' < \eta_1$ and $\eta_2' < \eta_2$ which may be chosen arbitrary close to η_1 and η_2, respectively. The kernel $T(\lambda, \mu, z)$ has limits

$$T(\lambda, \mu, E \pm i0), \quad E \in \Delta \setminus \sigma_p(H).$$

In our case, these limits are smooth in $\lambda, \mu \in \Delta \setminus \sigma_p(H)$. The scattering matrix for the pair (H_0, H) reads as

$$S_+(E) = I_{\mathfrak{h}} - 2\pi \mathrm{i}\, T(E, E, E + \mathrm{i}0), \quad E \in (a, b) \setminus \sigma_p(H).$$

Due to requirements (1.4) and (2.3) the point spectrum $\sigma_p(H)$ of H represents a finite set of eigenvalues with finite multiplicities (see Ref. [4]; cf. Ref. [12]).

Recall that the $T(\lambda, \mu, z)$ satisfies the following two Lippmann–Schwinger equations:

$$T(\lambda, \mu, z) = V(\lambda, \mu) - \int_a^b d\nu \, \frac{V(\lambda, \nu)\, T(\nu, \mu, z)}{\nu - z}, \tag{2.6}$$

$$T(\lambda, \mu, z) = V(\lambda, \mu) - \int_a^b d\nu \, \frac{T(\lambda, \nu, z)\, V(\nu, \mu)}{\nu - z}, \tag{2.7}$$

$$z \notin (a, b), \quad \lambda, \mu \in (a, b)$$

Substituting $T(\nu, \mu, z)$ in the r.h.s. part of Eq. (2.6) by the r.h.s. part of Eq. (2.7), one obtains

$$T(\lambda, \mu, z) = V(\lambda, \mu) - \int_a^b d\nu \, \frac{V(\lambda, \nu)\, V(\nu, \mu)}{\nu - z}$$
$$+ \int_a^b d\nu_1 \int_a^b d\nu_2 \, \frac{V(\lambda, \nu_1)\, T(\nu_1, \nu_2, z)\, V(\nu_2, \mu)}{(\nu_1 - z)(\nu_2 - z)}, \qquad z \notin [a, b]. \tag{2.8}$$

Since $V(\lambda, \mu)$ is analytic in both $\lambda, \mu \in \Omega$, one easily concludes from Eq. (2.8) that the kernel $T(\lambda, \mu, z)$ possesses the same holomorphy property. More detail statement is as follows.

Proposition 2.1. *If $z \notin (a,b) \cup \sigma_p(H)$, the function $T(\lambda,\mu,z)$ is holomorphic in both $\lambda \in \Omega$ and $\mu \in \Omega$. One can replace the interval (a,b) in Eqs.* (2.6) *and* (2.7) *by an arbitrary piecewise smooth Jordan contour $\gamma \subset \Omega$ obtained from (a,b) by continuous transformation provided that the end points are fixed and the point z is avoided during the transformation $(a,b) \to \gamma$.*

For the sake of simplicity, in the following we usually assume that the numbers $a,b \in \mathbb{R}$ are finite.

Now consider a smooth Jordan contour $\gamma \subset \Omega \cap \mathbb{C}^{\pm}$ obtained from the interval (a,b) by a continuous transformation with the fixed end points a and b. From the Proposition 2.1 it follows that Eq. (2.6) can be equivalently written as

$$T(\lambda,\mu,z) = V(\lambda,\mu) - \int_{\gamma} d\nu\, \frac{V(\lambda,\nu)\,T(\nu,\mu,z)}{\nu - z}, \tag{2.9}$$

$$\lambda,\mu \in \Omega, \quad z \in \mathbb{C} \setminus \Omega_{\gamma},$$

where the set $\Omega_{\gamma} \subset \mathbb{C}$ is confined by (and containing) the segment $[a,b]$ and the curve γ (see Fig. 1). By applying to Eq. (2.9) the Faddeev's approach of Ref. [4], one can prove that a solution $T(\lambda,\mu,z)$ exists and is analytic on z for any

$$z \notin \sigma_p(H) \cup \overline{\gamma} \cup \sigma_{\mathrm{res}}(\gamma), \tag{2.10}$$

where $\sigma_{\mathrm{res}}(\gamma)$ is a discrete set located inside Ω_{γ}; the overlining in $\overline{\gamma}$ means the closure, that is, $\overline{\gamma} = \gamma \cup \{a\} \cup \{b\}$. Because of the holomorphy of $V(\lambda,\mu)$ in $\lambda,\mu \in \Omega$, the solution $T(\lambda,\mu,z)$ remains analytic in $\lambda,\mu \in \Omega$ for any $z \in \mathbb{C}$ satisfying (2.10). The points of $\sigma_{\mathrm{res}}(\gamma)$ (resonances) correspond to the poles of the solution $T(z)$, which residues are finite rank operators. Hence, Eq. (2.9) allows one to pull the argument z of $T(z)$ from $\mathbb{C}^+$ to $\mathbb{C}^-$ at least into the interior of the set Ω_{γ}. Of course, the points of $\sigma_{\mathrm{res}}(\gamma)$ should be avoided during this procedure.

It turns out, however, that, after such a continuation, the solution $T(\lambda,\mu,z)$ for $z \in \Omega \cap \mathbb{C}^-$ is taken on an unphysical sheet of the Riemann energy surface of T. This unphysical sheet is attached to the physical sheet along the upper rim of the cut of $\mathbb{C}$ through the interval (a,b) and we denote it by Π_-. Thus, it is necessary to use a different notation, say, $T'(\lambda,\mu,z)$ for the continuation of the kernel of T on Π_- (in order to distinguish if from $T(z)$ at the same z on the physical energy sheet). By the

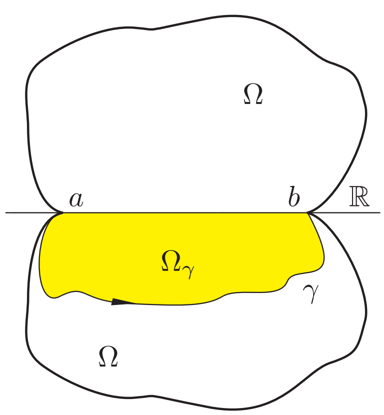

Figure 1: Domain Ω where the kernel $V(\lambda,\mu)$ is holomorphic both in λ and μ. The set Ω_{γ} is bounded by (and contains both) the Jordan contour γ and the segment $[a,b]$.

way, this kernel will coincide with the original one, that is, $T'(\lambda,\mu,z) = T(\lambda,\mu,z)$, provided $z \in \mathbb{C} \setminus \big(\Omega_\gamma \cup \sigma_p(H)\big)$.

The amazing thing is that the continued equation (2.9) may be solved explicitly. To show this, let us assume that $z \in \Omega_\gamma \setminus \big(\overline{\gamma} \cup \sigma_{\rm res}(\gamma)\big)$ and perform a two-step transformation of the contour γ (see Fig. 1) in the way shown in Fig. 2.

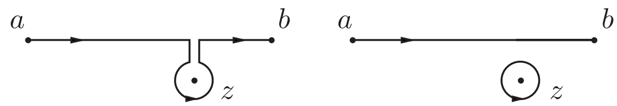

Figure 2: Two steps in transformation of the contour γ back to (a,b).

By performing such a transformation and computing the residue at $\nu = z$, one obtains from Eq. (2.9) the following equation for the unphysical-sheet values $T'(\lambda,\mu,z)$ of T:

$$T'(\lambda,\mu,z) = V(\lambda,\mu) - 2\pi{\rm i}\, V(\lambda,z)\, T'(z,\mu,z) - \int_a^b d\nu\, \frac{V(\lambda,\nu)T'(\nu,\mu,z)}{\nu - z}, \qquad (2.11)$$

$$\lambda,\mu \in \Omega, \qquad z \in \Omega \cap \mathbb{C}^-.$$

Adopting the standard terminology of scattering theory one calls the kernel $T'(z,\mu,z)$ "half-on-shell" since its first argument equals the spectral parameter (energy) z. Similarly, the kernel $T'(z,z,z)$ is called "(completely) on-shell", whereas the kernel $T'(\lambda,\mu,z)$ with arbitrary admissible values of λ and μ is called "off-shell". Surely, the adjectives "off-shell", "half-on-shell", and "on-shell" may be applied to any function of the complex arguments λ, μ, and z.

By transferring all the entries in Eq. (2.11) with the off-shell kernel T' to the l.h.s. one obtains:

$$T'(\lambda,\mu,z) + \int_a^b d\nu\, \frac{V(\lambda,\nu)\, T'(\nu,\mu,z)}{\nu - z} = V(\lambda,\mu) - 2\pi{\rm i}\, V(\lambda,z)\, T'(z,\mu,z), \qquad (2.12)$$

$$\lambda,\mu \in \Omega, \qquad z \in \Omega \cap \mathbb{C}^-.$$

Meanwhile, for z on the physical sheet we have: $\big(I + VR_0(z)\big)^{-1} V = T(z)$, $z \notin \sigma_p(H)$. This allows to rewrite Eq. (2.12) in the form

$$T'(\lambda,\mu,z) = T(\lambda,\mu,z) - 2\pi{\rm i}\, T(\lambda,z,z)\, T'(z,\mu,z), \qquad (2.13)$$

where the absence of the the prime in notation of the entry $T(\cdot,\mu,z)$ means that this entry is taken for z on the physical energy sheet. Now by setting $\lambda = z$ in Eq. (2.13), one gets

$$T'(z,\mu,z) = T(z,\mu,z) - 2\pi{\rm i}\, T(z,z,z)\, T'(z,\mu,z). \qquad (2.14)$$

From Eq. (2.14) it follows that

$$S_-(z)\, T'(z,\mu,z) = T(z,\mu,z), \qquad (2.15)$$

where

$$S_-(z) := I_{\mathfrak{h}} + 2\pi{\rm i}\, T(z,z,z), \qquad z \in \Omega \cap \mathbb{C}^-, \qquad (2.16)$$

is just the scattering matrix for the values of z in the lower half-plane. We emphasize that the values of z in Eq. (2.16) are taken on the physical sheet. From Eq. (2.15) it follows that

$$T'(z,\mu,z) = S_-(z)^{-1}\,T(z,\mu,z), \tag{2.17}$$

of course, under the condition that the inverse $S_-(z)^{-1}$ exists. That is, z in Eq. (2.17) should be such that $S_-(z)$ does not have eigenvalue zero. Combining Eqs. (2.13) and (2.17), one finally obtains

$$T'(\lambda,\mu,z) = T(\lambda,\mu,z) - 2\pi\mathrm{i}\,T(\lambda,z,z)\,S_-(z)^{-1}\,T(z,\mu,z). \tag{2.18}$$

All the entries on the r.h.s. part of Eq. (2.18) are taken for the same z as on the l.h.s. part but on the physical sheet. Thus, the representation (2.18) discloses the structure of the analytically continued transition operator $T'(z) = T(z)|_{\Pi_-}$ on the unphysical sheet Π_- exclusively in terms of the physical sheet.

An analytic continuation of $T(\lambda,\mu,z)$ from the lower half-plane $\mathbb{C}^-$ to the part $\Omega\cap\mathbb{C}^+$ of the unphysical energy sheet Π_+ attached to the physical sheet along the lower rim of the cut (a,b) may be performed exactly in the same way. As a result, one arrives at the following statement that works for both sheets Π_ℓ where the number $\ell=\pm1$ in the subscript is identified with the corresponding sign $\pm$ in the previous notation $\Pi_\pm$.

Lemma 2.2. *The transition operator $T(z)$ admits a meromorphic continuation (as an operator-valued function of the energy z) through the cut along the interval (a,b) both from the upper, $\mathbb{C}^+$, and lower, $\mathbb{C}^-$, half-planes to the respective parts*

$$\Omega_- := \Omega\cap\mathbb{C}^- \quad \textit{and} \quad \Omega_+ := \Omega\cap\mathbb{C}^+$$

of the unphysical sheets Π_{-1} and Π_{+1} attached to the physical sheet along the upper and lower rims of the above cut. The kernel of the continued operator $T(z)\big|_{\Pi_\ell\cap\Omega_\ell}$, $\ell=\pm1$, is given by the equality

$$T(\lambda,\mu,z)\big|_{z\,\in\,\Pi_\ell\,\cap\,\Omega_\ell} = \big(T(\lambda,\mu,z) + 2\pi\mathrm{i}\,\ell\,T(\lambda,z,z)\,S_\ell(z)^{-1}\,T(z,\mu,z)\big)\big|_{z\,\in\,\Omega_\ell}, \tag{2.19}$$

$$z\in\Omega_\ell\setminus\sigma^\ell_{\mathrm{res}}, \tag{2.20}$$

with all the entries on the r.h.s. part, including the scattering matrix

$$S_\ell(z) = I_{\mathfrak{h}} - 2\pi\mathrm{i}\,\ell\,T(z,z,z), \tag{2.21}$$

being taken for the same z on the physical sheet. Notation $\sigma^\ell_{\mathrm{res}}$ is used for the set of all those points $\zeta\in\Omega\cap\mathbb{C}^\ell$ where $S_\ell(\zeta)$ has eigenvalue zero.

It is worth mentioning that some further analytic properties of $V(\lambda,\mu)$ outside Ω should be known in order to decide whether Π_- and Π_+ represent the same ("second") unphysical sheet or they are really different sheets of the energy Riemann surface (cf. Ref. [14]).

Continuation formula for the scattering matrix is a simple corollary to Lemma 2.2.

Lemma 2.3. *An analytic continuation of the scattering matrix $S_{-\ell}(z)$, $\ell=\pm1$, to the unphysical sheet Π_ℓ is is given by*

$$S_{-\ell}(z)\big|_{z\in\Pi_\ell\cap\Omega_\ell} = S_\ell(z)^{-1}\big|_{z\in\Omega_\ell}, \qquad z\notin\sigma^\ell_{\mathrm{res}}, \tag{2.22}$$

where the r.h.s. part is taken for z on the physical sheet.

3 Equivalence of the deformation and scattering resonances in the Friedrichs–Faddeev model

From now on we consider a family of the Friedrichs–Faddeev Hamiltonians

$$H_\gamma = H_{0,\gamma} + V_\gamma$$

associated with smooth Jordan curves $\gamma \subset \Omega$ obtained by continuous transformation from the interval (a, b), with the end points a, b fixed during the transformation. As before, the notation Ω is used for the holomorphy domain of $V(\lambda, \mu)$ in λ, μ. The domain Ω may or may not include the points a and/or b. The entries $H_{0,\gamma}$ and V_γ are given by

$$\big(H_{0,\gamma} f\big)(\lambda) = \lambda\, f(\lambda) \qquad \text{and} \qquad \big(V_\gamma f\big)(\lambda) = \int_\gamma V(\lambda, \mu)\, f(\mu)\, d\mu, \qquad \lambda \in \gamma.$$

It is assumed that $f \in L_2(\gamma, \mathfrak{h})$ where $L_2(\gamma, \mathfrak{h})$ is the Hilbert space of $\mathfrak{h}$-valued functions of the variable $\lambda \in \gamma$ with the scalar product

$$\langle f, g\rangle_\gamma = \int_\gamma |d\lambda| \langle f(\lambda), g(\lambda)\rangle_{\mathfrak{h}}.$$

Notice once again that the standard complex scaling [16, 17] of a two-body Hamiltonian may be viewed as a particular case of the complex deformation of the Friedrichs–Faddeev model (see Ref. [6, Section 3]).

Assume, for simplicity, like in Section 2, that both a and b are finite real numbers and let $V(\lambda, \mu)$ be also as in Section 2. As usually, for the resolvent $R_\gamma(z) = (H_\gamma - z)^{-1}$ of the operator H_γ we have

$$R_\gamma(z) = R_{0,\gamma}(z) - R_{0,\gamma}(z)\, T_\gamma(z)\, R_{0,\gamma}(z), \tag{3.1}$$

where $R_{0,\gamma}(z)$ is the resolvent of $H_{0,\gamma}$,

$$R_{0,\gamma}(z) = (H_{0,\gamma} - z)^{-1}, \qquad z \notin \sigma(H_{0,\gamma}),$$

and

$$T_\gamma(z) = V_\gamma - V_\gamma\,(H_\gamma - z)^{-1}\, V_\gamma, \qquad z \notin \sigma(H_\gamma) \tag{3.2}$$

is the transition operator for the pair $(H_{0,\gamma}, H_\gamma)$.

Clearly, $H_{0,\gamma}$ is a normal operator on $\mathfrak{H}_\gamma$. Its spectrum is purely absolutely continuous and fills the curve $\overline{\gamma}$. From Eq. (3.1) it immediately follows that the discrete eigenvalues of H_γ are associated just with the poles of the operator-valued function $T_\gamma(z)$.

Suppose that the above Jordan contour γ lies entirely in $\Omega_- = \Omega \cap \mathbb{C}^-$ (or entirely in $\Omega_+ = \Omega \cap \mathbb{C}^+$) and let Ω_γ be again the set in $\mathbb{C}$ confined by (and containing) the interval $[a, b]$ and the curve γ (see Fig. 1).

Lemma 3.1. *The following equality holds: $\sigma(H_\gamma) \setminus \Omega_\gamma = \sigma_p(H) \setminus \overline{\Delta}$, which means that the spectrum of H_γ outside Ω_γ is purely real and coincides with the corresponding eigenvalue set of H. Furthermore, $\sigma_p(H_\gamma) \cap \Delta = \sigma_p(H) \cap \Delta$, i. e., the eigenvalues of H_γ lying on Δ do not depend on the (smooth) Jordan contour γ. Finally, the spectrum of H_γ inside Ω_γ consists of the scattering-matrix resonances.*

We skip a detail justification of this assertion and refer the reader to the proof of the corresponding statement in Ref. [6, Proposition 4.1]. Here we only notice that the proof in Ref. [6] is reduced to the observation that the kernels of the T-matrices (3.2) and (2.4) possess the property

$$T_\gamma(\lambda,\mu,z) = T(\lambda,\mu,z) \quad \text{whenever} \quad \lambda,\mu\in\gamma,\ z\in\mathbb{C}\setminus\Omega_\gamma \quad (\text{and } z\notin\sigma_p(H)).$$

Then, by the uniqueness principle for the analytic continuation, one concludes that, for z inside Ω_γ, the kernel $T_\gamma(\lambda,\mu,z)$ represents just the analytic continuation of $T(\lambda,\mu,\cdot)$ to the interior of Ω_γ belonging already to the unphysical sheet. Hence, the poles of $T_\gamma(z)$ within Ω_γ represent resonances of the original Friedrichs–Faddeev Hamiltonian $\big($the one associated with the interval $(a,b)\big)$. This also means that the positions of the resonances inside Ω_γ are stable in the sense that they do not depend on γ.

Conclusion

In this work we have studied the Friedrichs–Faddeev model with an analytic potential kernel $V(\lambda,\mu)$. We have found that the transition operator and the scattering matrix for this model, analytically continued on unphysical energy sheets, admit explicit representations in terms of the same operators considered exclusively on the physical sheet. A resonance on the unphysical sheet Π_ℓ, $\ell=\pm1$, or, more precisely, in the domain $\Pi_\ell\cap\Omega_\ell$, is a point, for the copy z of which on the physical sheet the scattering matrix $S_\ell(z)$ has eigenvalue zero, i. e.,

$$S_\ell(z)\,\mathcal{A}=0 \quad \text{for a non-zero} \quad \mathcal{A}\in\mathfrak{h}.$$

We have also shown that, for the Friedrichs–Faddeev model under consideration, the deformation resonances are nothing else but the scattering matrix resonances.

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