Three-Nucleon Calculations within the Bethe–Salpeter Approach with Rank-One Separable Kernel

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Abstract

Relativistic properties of a three-nucleon system are investigated using the Bethe–Salpeter approach. A system of integral Faddeev-type equations for the three-particle system amplitudes is obtained. The nucleon-nucleon interaction is chosen to be in a separable form. The Gauss quadrature method for solving the integral system of equations is considered. The binding energy and the partial-wave amplitudes ($^1S_0$ and $^3S_1$) of the triton are found by solving the system of the integral equations.

Keywords: *Bethe–Salpeter approach; three-nucleon system; Faddeev equation*

Introduction

Three-body calculations in nuclear physics are very interesting for describing three-nucleon bound states (^{3}He, T), processes of elastic, inelastic and deep inelastic scattering of leptons by light nuclei and also the hadron-deuteron reactions (for example, $pd \to pd$, $pd \to ppn$). The study of the ^{3}He and T nuclei is also interesting because it allows us to investigate a further (in addition to the deuteron) evolution of a bound nucleon thereby contributing to the explanation of the so-called EMC-effect.

Faddeev equations are used in quantum mechanics to describe three-particle systems. The main feature of Faddeev equations is that all three particles interact through a pair potential. We are interested in reactions at high momentum transfer where the relativistic methods should be used. One of such methods is the Bethe–Salpeter (BS) approach. The relativistic analog of the Faddeev equations can be considered in the BS formalism.

In this paper, all nucleons have equal masses. The scalar propagators instead of the spinor ones are used also for simplicity. The spin and isospin structure of the nucleons is taken into account by using the so-called recoupling-coefficient matrix. The work mainly follows the ideas of Ref. [1].

The paper is organized as follows. A two-particle problem is considered in Section 1 and Section 2 is devoted to three-particle equations. In Section 3 we present the calculations and results. The summary is given in Section 4.

1 Two-particle case

Since the formalism of the Faddeev equations is based on properties of the pair nucleon-nucleon interaction, here only some conclusions from the two-body problem

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http://www.ntse-2014.khb.ru/Proc/Yurev.pdf.

Table 1: Parameters λ and β for the $^1S_0$ and $^3S_1$ partial-wave states.

	$^1S_0$	$^3S_1$
λ (GeV^4)	-1.12087	-3.15480
β (GeV)	0.287614	0.279731

are given.

The Bethe–Salpeter equation for a relativistic two-particle system is written in the following form:

$$T(p,p';s) = V(p,p') + \frac{i}{4\pi^3}\int d^4k\, V(p,k)\, G(k;s)\, T(k,p';s), \tag{1}$$

where $T(p,p';s)$ is the two-particle T matrix and $V(p,p')$ is the kernel (potential) of the nucleon-nucleon interaction. The free two-particle Green's function $G(k;s)$ is expressed, for simplicity, thought the scalar propagator of the nucleons:

$$G^{-1}(k;s) = \left[(P/2+k)^2 - m_N^2 + i\epsilon\right]\left[(P/2-k)^2 - m_N^2 + i\epsilon\right]. \tag{2}$$

To solve Eq. (1), the separable *ansatz* for the nucleon-nucleon potential $V(p,p')$ is used (rank-one):

$$V(p_0,p,p_0',p') = \lambda\, g(p_0,p)\, g(p_0',p'). \tag{3}$$

In this case the two-particle T matrix has the following simple form:

$$T(p_0,p,p_0',p';s) = \tau(s)\, g(p_0,p)\, g(p_0',p'), \tag{4}$$

where

$$\tau(s) = \left[\lambda^{-1} - \frac{i}{4\pi^3}\int_{-\infty}^{\infty} dk^0 \int_0^{\infty} k^2 dk\, g^2(k^0,k)\, G(k^0,k;s)\right]^{-1}. \tag{5}$$

As the simplest assumption, the relativistic *Yamaguchi*-type form factor $g_Y(p_0,p)$ is used,

$$g_Y(p_0,p) = \frac{1}{-p_0^2 + p^2 + \beta^2}, \tag{6}$$

with parameters λ and β chosen to describe the experimental data. The values of the parameters are given in Table 1.

To calculate the scattering phase shifts of proton-neutron elastic collisions, the following parametrization of the on-mass-shell T matrix is used:

$$T_L(\bar{p}) = T_L(0,\bar{p},0,\bar{p};s) = \frac{-8\pi\sqrt{s}}{\bar{p}}\, e^{i\delta_L(\bar{p})} \sin\delta_L(\bar{p})$$

with $\delta_L(\bar{p})$ being the scattering phase shifts and $\bar{p} = \sqrt{s/4 - m_N^2} = \sqrt{\frac{1}{2} m_N T_{lab}}$. The calculated scattering phase shifts together with the experimental data are shown in Fig. 1. The results of calculations of the low-energy parameters and properties of the bound state (deuteron) are given in Table 2 together with the experimental data from Ref. [3].

As it seen from Table 2, the properties of low-energy proton-neutron scattering in the $^1S_0$ and $^3S_1$ partial waves and the deuteron binding are in a satisfactory agreement with the experimental data. However, as is seen in Fig. 1, the scattering phase shifts describe the experiment up to $T_{lab} = 100{-}120$ MeV only. This disadvantage is due to the simplest rank-one choice of the nucleon-nucleon kernel.

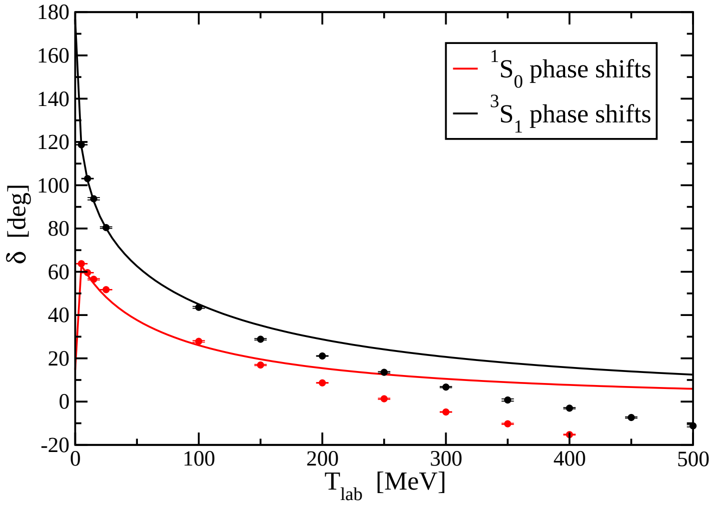

Figure 1: Scattering phase shifts for the relativistic Yamaguchi-type form factors. The experimental data are taken from Ref. [2]

Table 2: The scattering length a_0, the effective range r_0 and the deuteron binding energy E_d for the $^1S_0$ and $^3S_1$ partial waves.

	$^3S_1$	experiment	$^1S_0$	experiment
a_0 (fm)	5.424	5.424(4)	−23.748	−23.748(10)
r_0 (fm)	1.775	1.759(5)	2.75	2.75(5)
E_d (MeV)	2.2246	2.224644(46)		

2 Three-particle case

A three-particle system can be described by the Faddeev equations

$$\begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} - \begin{bmatrix} 0 & T_1G_1 & T_1G_1 \\ T_2G_2 & 0 & T_2G_2 \\ T_3G_3 & T_3G_3 & 0 \end{bmatrix} \begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix}, \tag{7}$$

where the full matrix $T = \sum_{i=1}^{3} T^{(i)}$, G_i is the two-particle (j and n) Green's function [ijn is cyclic permutation of (1,2,3)],

$$G_i(k_j, k_n) = \frac{1}{(k_j^2 - m_N^2 + i\epsilon)(k_n^2 - m_N^2 + i\epsilon)}, \tag{8}$$

and T_i is the two-particle T matrix which can be written as

$$T_i(k_1, k_2, k_3; k_1', k_2', k_3') = (2\pi)^4\, \delta^{(4)}(K_i - K_i')\, T_i(k_j, k_n; k_j', k_n'). \tag{9}$$

For the system of equal-mass particles, the Jacobi momenta can be written as

$$p_i = \frac{1}{2}(k_j - k_n), \quad q_i = \frac{1}{3}K - k_i, \quad K = k_1 + k_2 + k_3. \tag{10}$$

Using expressions (10), Eq. (7) can be rewritten as

$$T^{(i)}(p_i, q_i; p_i', q_i'; s) = (2\pi)^4\, \delta^{(4)}(q_i - q_i')\, T_i(p_i; p_i'; s)$$

$$- i \int \frac{dp_i''}{(2\pi)^4}\, T_i(p_i; p_i''; s)\, G_i(k_j'', k_n'') \left[T^{(j)}(p_j'', q_i''; p_i', q_i'; s) + T^{(n)}(p_i'', q_i''; p_i', q_i'; s)\right]. \tag{11}$$

For the three-particle bound state it is suitable to introduce an amplitude

$$\Psi^{(i)}(p_i, q_i; s) = \langle p_i, q_i | T^{(i)} | M_B \rangle, \tag{12}$$

where $M_B = \sqrt{s} = 3m_N - E_B$ is the mass of the bound state (triton) and $s = K^2$ is the total momentum squared. Assuming the orbital angular momenta in the triton to be equal to zero ($l_p = l_q = 0$), only two partial-wave states ($^1S_0$ and $^3S_1$) should be taken into account. In the case of the two-particle T matrix in the separable form (4), the amplitude of the triton becomes

$$\Psi^{(i)}(p, q; s) = \sum_{j=1,2} g_j(p)\, \tau_j(s)\, \Phi_j(q; s), \tag{13}$$

where $j = 1(^1S_0), 2(^3S_1)$. The functions $\Phi_j(q)$ satisfy the following system of integral equations:

$$\Phi_j(q_0, q; s) = \sum_{j'} \frac{i}{4\pi^3} \int dq_0' \int {q'}^2 dq'$$

$$\times\, Z_{jj'}(q_0, q, q_0', q'; s)\, \frac{\tau_{j'}\left[(\frac{2}{3}\sqrt{s} + q_0')^2 - {q'}^2\right]}{\left(\frac{1}{3}\sqrt{s} - q_0\right)^2 - {q'}^2 - m_N^2 + i\epsilon}\, \Phi_{j'}(q_0', q'; s). \tag{14}$$

The so-called effective energy-dependent potential Z is

$$Z_{jj'}(q_0, q, q_0', q'; s) = C_{jj'} \int_{-1}^{1} d(cos\vartheta_{qq'})$$

$$\times\, \frac{g_j\left(-\frac{1}{2}q_0 - q_0', |-\frac{1}{2}\mathbf{q} - \mathbf{q}'|\right) g_{jj'}\left(q_0 + \frac{1}{2}q_0', |\mathbf{q} + \frac{1}{2}\mathbf{q}'|\right)}{\left(\frac{1}{3}\sqrt{s} + q_0 + q_0'\right)^2 - (\mathbf{q} + \mathbf{q}')^2 - m_N^2 + i\epsilon}, \tag{15}$$

where $C_{jj'}$ is the spin and isospin recoupling-coefficient matrix:

$$C_{jj'} = \begin{bmatrix} \frac{1}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{bmatrix}. \tag{16}$$

The system of integral equations (14)–(15) has a number of singularities, namely:

- poles from the one-particle propagator:

$$q'^0_{1,2} = \frac{1}{3}\sqrt{s} \mp [E_{|\mathbf{q}'|} - i\epsilon];$$

- poles from the propagator in the Z-function:

$$q'^0_{3,4} = -\frac{1}{3}\sqrt{s} - q^0 \pm [E_{|\mathbf{q}'+\mathbf{q}|} - i\epsilon];$$

- poles from the Yamaguchi-functions:

$$q'^0_{5,6} = -2q^0 \pm 2[E_{|\frac{1}{2}\mathbf{q}'+\mathbf{q}|,\beta} - i\epsilon]$$

  and

$$q'^0_{7,8} = -\frac{1}{2}q^0 \pm \frac{1}{2}[E_{|\mathbf{q}'+\frac{1}{2}\mathbf{q}|,\beta} - i\epsilon];$$

- cuts from the two-particle propagator τ:

$$q'^0_{9,10} = \pm\sqrt{q'^2 + 4m_N^2} - \frac{2}{3}\sqrt{s} \quad \text{and} \quad \pm\infty;$$

- poles from the two-particle propagator τ:

$$q'^0_{11,12} = \pm\sqrt{q'^2 + 4M_d^2} - \frac{2}{3}\sqrt{s}.$$

However in the case of the bound three-particle system ($\sqrt{s} \leq 3m_N$), all above singularities do not cross the path of integration over q_0 and thus do not affect the Wick rotation procedure $q_0 \to iq_4$. Therefore Eqs. (14)–(15) become:

$$\Phi_j(q_4, q; s) = -\frac{1}{4\pi^3} \sum_{j'=1}^{2} \int_{-\infty}^{\infty} dq'_4 \int_0^{\infty} q'^2 dq' \times Z_{jj'}(iq_4, q; iq'_4, q'; s)\, \frac{\tau_{j'}\left[(\frac{2}{3}\sqrt{s} + iq'_4)^2 - q'^2\right]}{\left(\frac{1}{3}\sqrt{s} - iq'_4\right)^2 - q'^2 - m_N^2}\, \Phi_{j'}(q'_4, q'; s) \quad (17)$$

and

$$Z_{jj'}(q_4, q; q'_4, q'; s) = C_{jj'} \int_{-1}^{1} d(cos\vartheta_{qq'}) \times \frac{g_j\left(-\frac{1}{2}q^0 - q^{0'}, |\frac{1}{2}\mathbf{q} + \mathbf{q}'|\right) g_j\left(q^0 + \frac{1}{2}q^{0'}, |\mathbf{q} + \frac{1}{2}\mathbf{q}'|\right)}{\left(\frac{1}{3}\sqrt{s} + q^0 + q^{0'}\right)^2 - (|\mathbf{q} + \mathbf{q}'|)^2 - m_N^2}. \quad (18)$$

Various methods can be used to solve Eqs. (17)–(18). One of them is discussed in the next section.

3 Solution and results

In order to solve the system of integral equations, the Gaussian quadrature method is used. The integration variables q $[0, \infty)$ and q_4 $(-\infty, \infty)$ are mapped to the $[-1, 1]$ interval. The quadrature method replaces integrals by sums and transforms the system of homogeneous linear integral equations to a system of algebraic equations. These equations can be solved using FORTRAN codes.

The method can be presented schematically as

$$f(x) = \int A(x, y)\, f(y) \to f(x_i) = \sum_{j=1,n} A(x_i, y_j)\, w_j\, f(y_j),$$

where x_i, y_j and w_j are the Gauss points and weights and n is the number of points. The homogeneous system of linear equations takes the form

$$M\phi = 0$$

with $M_{ij} \equiv A_{ij} - \delta_{ij}$ and $\phi_i = f(x_i)$; $i, j = 1, 2, ..., n$. This system of equations has a solution if the determinant of the matrix is equal to zero. This condition is satisfied at the binding energy of the three-nucleon system:

$$\det M(s) = 0 \quad \text{at} \quad \sqrt{s} = 3m_N - E_B.$$

The result of calculations ($n = 15$) for the binding energy is $E_B = 11.09$ MeV which should be compared to the experimental value of 8.48 MeV. The difference can be explained by the simplicity of the rank-one separable kernel of the nucleon-nucleon interaction.

The obtained partial-wave amplitudes are shown in Figs. 2–4. The imaginary parts of the amplitudes arise as a pure relativistic effect which does not appears in nonrelativistic Faddeev equations.

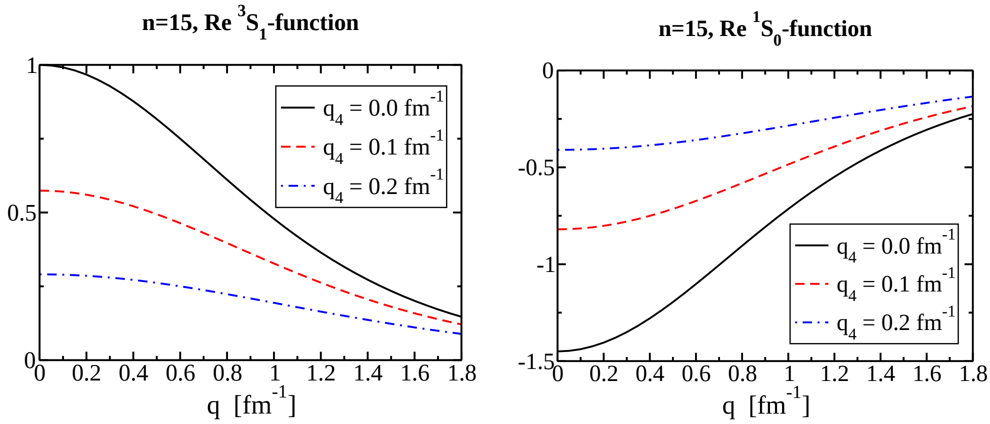

Figure 2: Real parts of the $^3S_1$ (left) and $^1S_0$ (right) amplitudes as functions of q at various q_4 values.

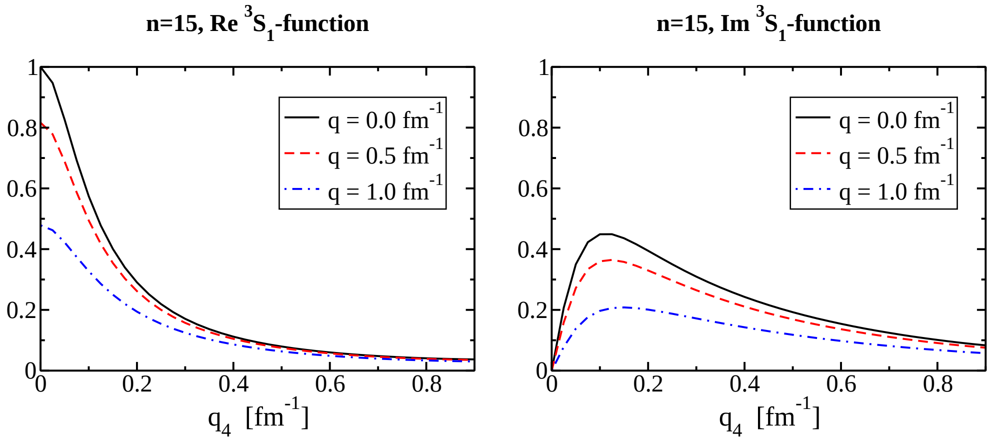

Figure 3: Real (left) and imaginary (right) parts of the $^3S_1$ amplitude as functions of q_4 at various q values.

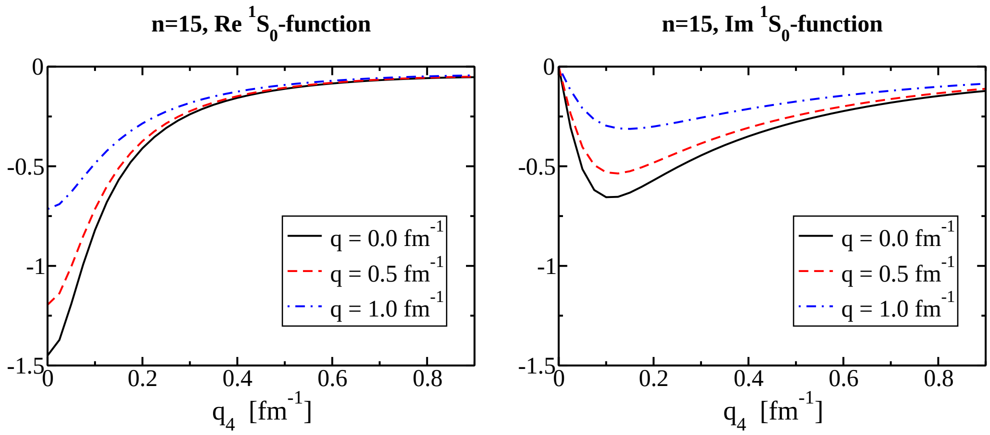

Figure 4: Real (left) and imaginary (right) parts of the $^1S_0$ amplitude as functions of q_4 at various q values.

4 Summary

In this paper a three-body system in the Bethe–Salpeter approach is investigated. A rank-one separable nucleon-nucleon interaction is utilized. The form factor is chosen in the form of a relativistic generalization of the Yamaguchi-type function. The parameters of the nucleon-nucleon potential in the $^1S_0$ and $^3S_1$ partial waves reproduce low-energy scattering parameters and deuteron properties as well as the phase shifts up to the laboratory energy of 100–120 MeV.

The Faddeev equations for the triton wave functions considered in the BS formalism are solved using the Gauss quadrature method. The triton binding energy and amplitudes of the $^1S_0$ and $^3S_1$ partial-wave states are calculated.

The triton binding energy is essentially overestimated. To improve the results, the rank of the separable kernel should be increased. Other partial waves, the P and D waves in particular, and the spinor propagators for nucleons should be also taken into account.

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