NTSE, Ames, USA

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Nonperturbative calculations in the Light-Front Field Theory

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• Outline

- Fock space and its truncation
- Wick-Cutkosky model.
- Yukawa Model
  - Two-body truncation.
  - Three-body truncation.
- E.M. form factors and anomalous magnetic moment.
- Conclusion.

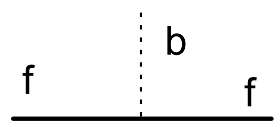
• Field theory

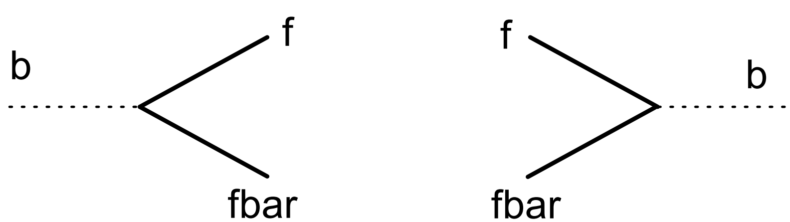
Main features:

- Infinite number degrees of freedom.
- The number of particles is not conserved.

Interaction: (fermion – spinless meson):

$$H^{int} = g\bar{\psi}\psi\Phi$$





• Approximating by finite d.o.f.

- Lattice calculations
  - Calculating large-dimensional integrals.
- Truncated Fock decomposition.
  - Solving large system of equations.

Both require supercomputers!
Truncated Fock decomposition. –For first non-trivial truncation has been already solved at laptop.
-Aim of this talk.

• State vector

The state vector is represented as the (exact) Fock decomposition:

$$|p\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \dots \\ \psi_N \\ \dots \end{pmatrix}$$

Approximation: replace this infinite column by the finite (truncated) one.
State vector is defined on LF and it is calculated in LFD.

An alternative to the lattice calculations?

• Eigenvalue equation:

$$H\,|p\rangle = M\,|p\rangle$$

It results in a system of equations for the Fock components ψ_n.

$$\begin{pmatrix} H^0_{11} & H^{int}_{12} & 0 & 0 \\ H^{int}_{21} & H^0_{22} & \dots & 0 \\ 0 & \dots & \dots & \dots \\ 0 & 0 & \dots & H^0_{NN} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \dots \\ \psi_N \end{pmatrix} = M \begin{pmatrix} \psi_1 \\ \psi_2 \\ \dots \\ \psi_N \end{pmatrix}$$

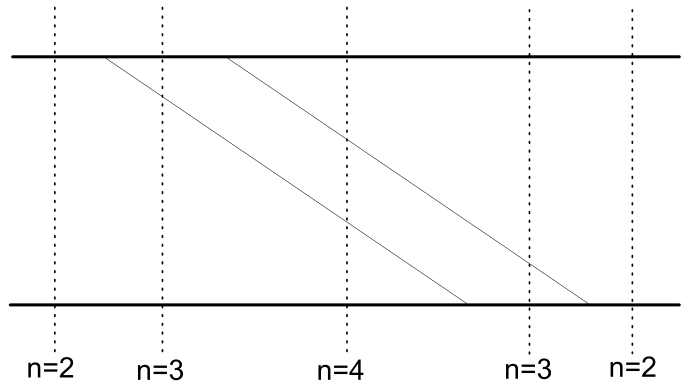
The coupling constant α in H^{int} may be large. After truncation, the numerical solution of the system of equations is non-perturbative.

• Can it converge enough fast?

This can be checked in a simple solvable model (Wick-Cutkosky model).

Dae Sung Hwang and V.A. Karmanov,
Many-body Fock sectors in Wick-Cutkosky model,
Nucl. Phys. **B696** (2004) 413.

• Wick-Cutkoski modeel



- Spinless particles, massless exchanges.
- The ladder graphs only (however, including all the stretched boxes). No self-energy, no divergences.
- However, many-body intermediate states, up to infinity, are taken into account.
- One can compare with truncated calculation.

• Normalization integral

$$\langle p|p\rangle = 1 = \int \psi_2^2 \ldots + \int \psi_3^2 \ldots + \int \psi_4^2 \ldots + \cdots = N_2 + N_3 + N_4 + \cdots$$

Take huge coupling constant: $\alpha = 2\pi$ (when $\alpha_{QED} = \frac{1}{137}$)

Then, 2- and 3-body sectors dominates:

N_2	N_3	$N_{n\geq 4}$	$N_2 + N_3 + N_{n\geq 4}$
9/14=64%	26%	10%	100%

The result is non-perturbative:

–Infinite series in terms of (large) coupling constant.

–Finite number of intermediate states.

Similar hierarchy takes place for e.m. form factors.

• Competition:

Coupling constant – energy

Coupling constant: $\alpha = \frac{g^2}{16\pi m^2}$.
n exchanged particles in intermediate state:

$$M \sim \frac{\alpha^n}{E - \sum_{i=1}^{n} E_i}$$

If $\alpha > 1$ $(\alpha \to 2\pi)$, then α^n is large.
But $\sum_{i=1}^{n} E_i$ is also large.

• The approach is promissing!

Being developed enough, it might form an alternative to the lattice calculations.

General advantage:
knowing state vector (LF wave functions), we can calculate any observable. – Advocated by S. Brodsky.

Particular profit:
Minkowski space, wave functions, form factors, etc.

Yukawa model plays role of a testing area.

• Yukawa model and QED

St. Glazek R. Perry	Yukawa model, 2-body truncation
S.J. Brodsky S. Chabysheva V.A. Franke J.R. Hiller G. McCartor S.A. Paston E.V. Prokhvatilov	Bare masses basis $m_0 \to m$
S. Chabysheva J.R. Hiller	Coupled-cluster method
J. Vary et al. (ISU)	Harmonic oscillator basis

• Explicitly covariant LFD

V.A. Karmanov, JETP, **44** (1976) 201.
J. Carbonell, B. Desplanques, V.A. Karmanov, J.-F. Mathiot, Phys. Reports, **300** (1998) 215.

$$t + z = 0 \quad \rightarrow \quad \omega{\cdot}x = \omega_0 t - \vec{\omega}{\cdot}\vec{x}$$

where $\omega = (\omega_0, \vec{\omega})$ such that $\omega^2 = 0$.

The unit vector $\vec{n} = \frac{\vec{\omega}}{|\vec{\omega}|}$ determines the orientation of the light-front plane.

Particular case: $\omega = (1, 0, 0, -1)$ corresponds to the standard approach.

• Yukawa Lagrangian

$$\mathcal{L} = \mathcal{L}^{free} + \mathcal{L}^{int}$$

Free Lagrangian:

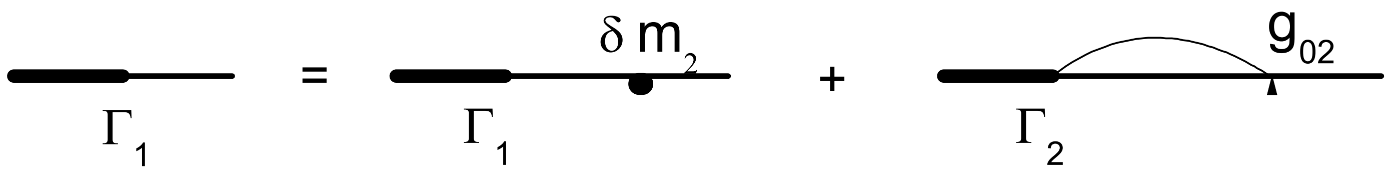
$$\mathcal{L}^{free} = i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi + \frac{1}{2}\left[\partial_\mu \Phi \partial^\mu \Phi - \mu^2 \Phi^2\right]$$

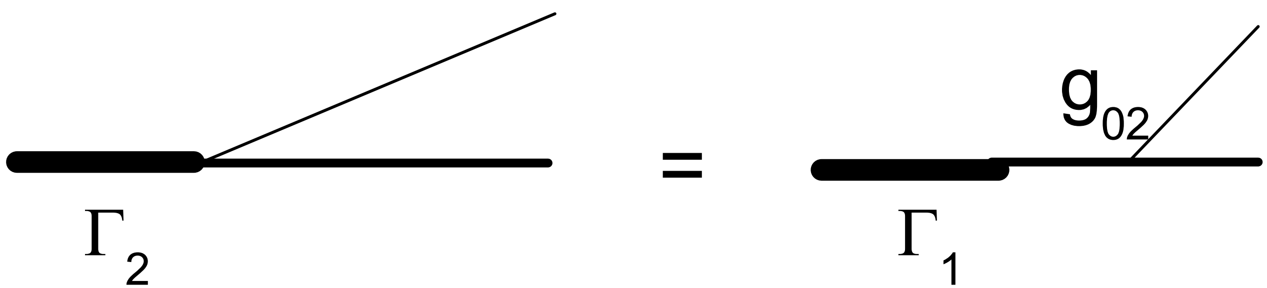
Interaction Lagrangian:

$$\mathcal{L}^{int} = g_0 \bar{\Psi}\Psi\Phi + \delta m \bar{\Psi}\Psi,$$

+ 1 PV fermion and 1 PV boson.

• Two-body truncation





System of equation for physical and Pauli-Villars particles (one PV fermion and one PV boson).

• Two-body wave function

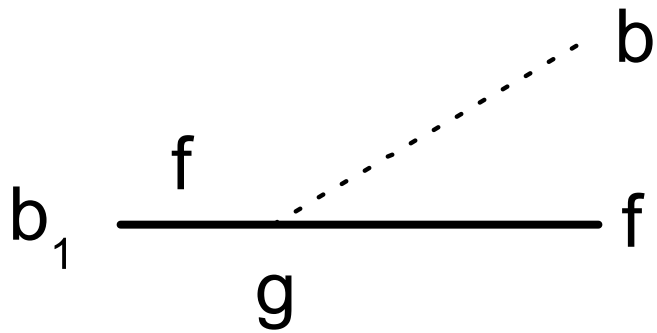
Spin structure

$$\bar{u}(k)\Gamma_2 u(p) = \bar{u}(k)\left[b_1 + \frac{m\not{\omega}}{\omega\cdot p} b_2\right] u(p)$$

Two spin components b_1, b_2.

For two-body truncation $b_1 = const$, $b_2 = 0$.

• Renormalization condition



$$b_1 = g$$

Since it is just the interaction vertex ffb.

δm_2 is found as an eigenvalue.

• Two-body solution

$$\psi_1 = \frac{1}{2m} \quad b_1 = g, \quad b_2 = 0.$$

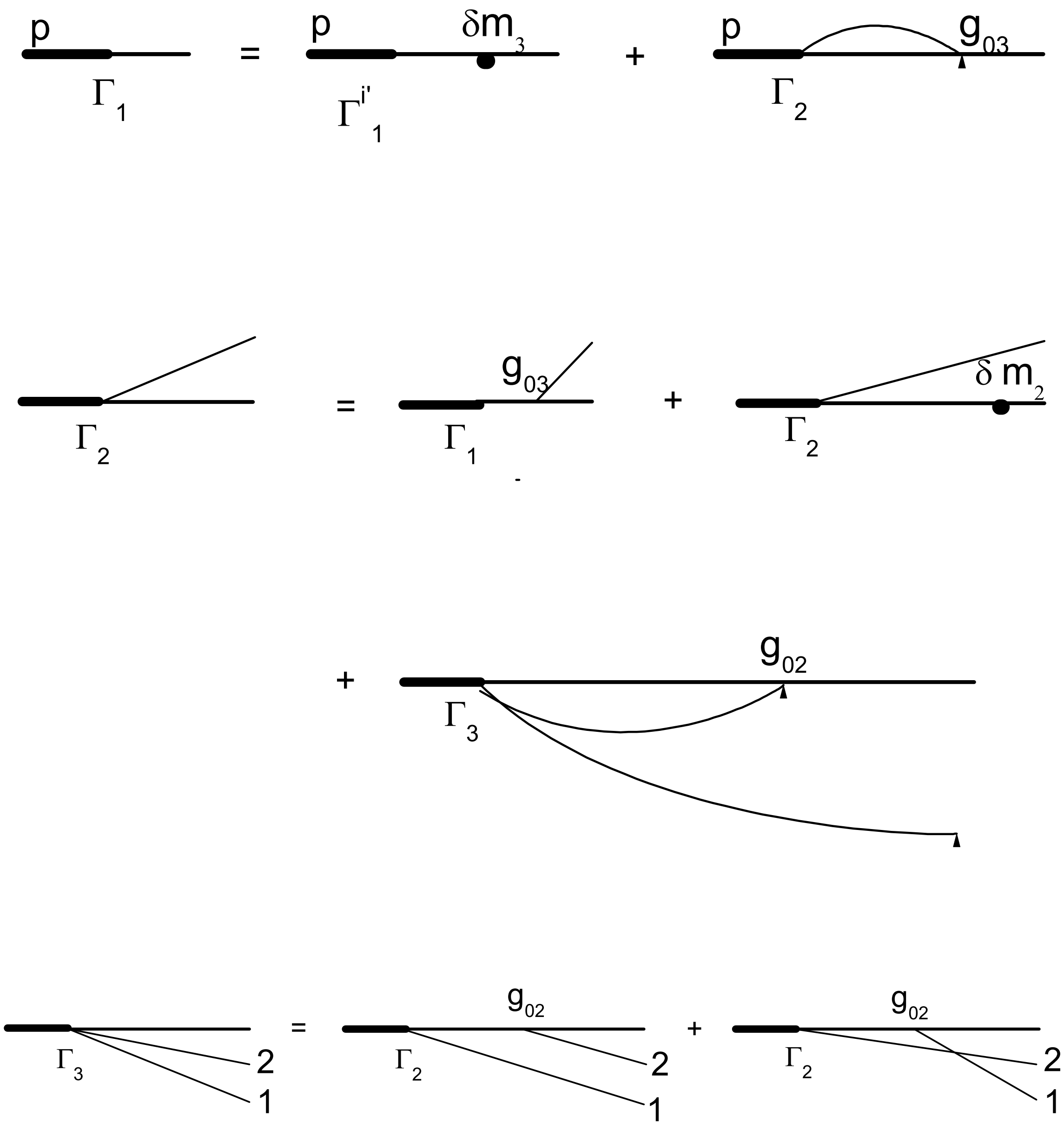
$$g_{02}^2 = \frac{g^2}{1 - I_2}, \quad \delta m_2 = g_{02}^2 \Sigma(\not p = m)$$

Comment:

$g_{02}, \delta m_2$: index "0" means "bare", index "2" means "found in the 2-body truncation".

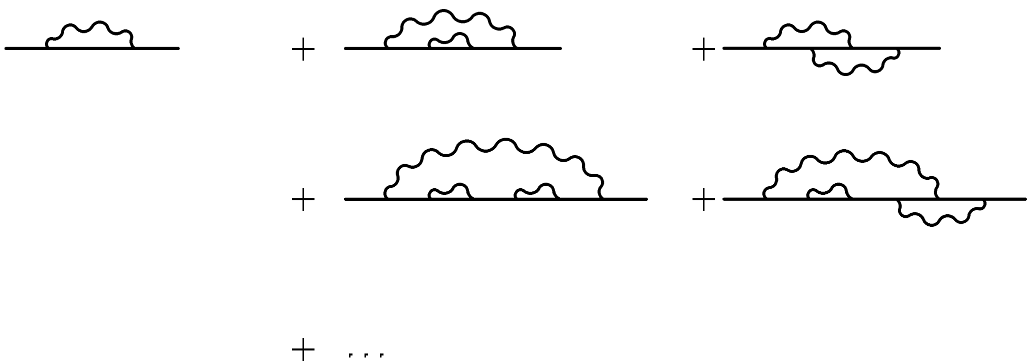
• Three-body truncation

System of equations



• Infinite set of irreducible graphs

Example: Three-body self-energy



Perturbative expansion, in terms of the pion-nucleon coupling constant g, of the nucleon self-energy.

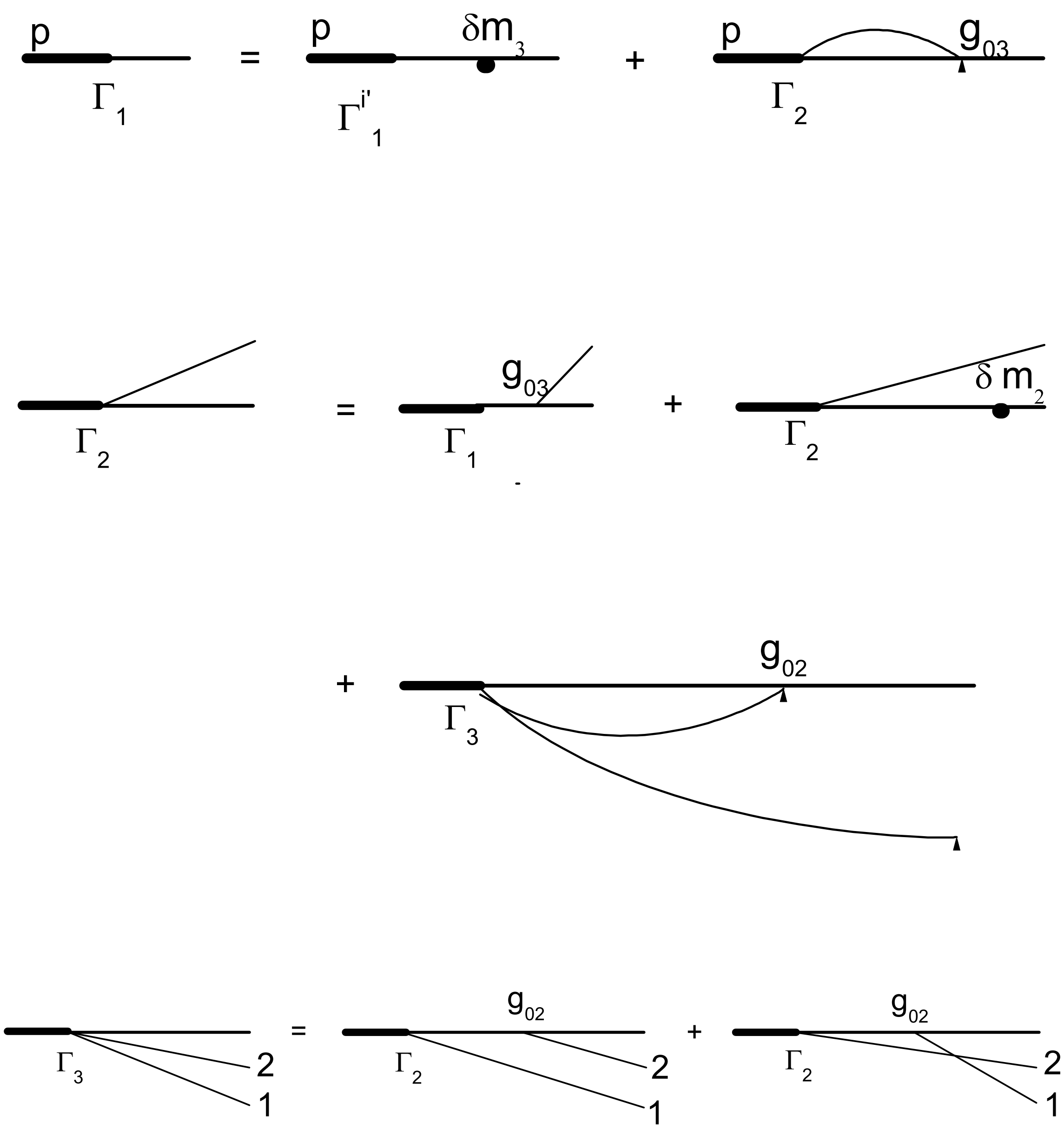
• Consequences

- All the orders of perturbation decomposition in the degrees of g.
- For given order of g – not full set of perturbative graphs.

Infinities, after renormalization, are not cancelled.

• Three-body truncation

System of equations



• Sector-dependent counter terms

To provide cancellations of infinities

R. Perry, A. Harindranath, K. Wilson,
Phys. Rev. Lett. 24 (1990) 2959.

Our practical realization of this scheme.

V.A. Karmanov, J.-F. Mathiot, A.V. Smirnov,
Phys. Rev. **D77** (2008) 085028.

• Determining counter terms

$\delta m_2, g_{02}$ are determined in two-body sector:

$$\delta m_2 = \Sigma(p = m), \quad g_{02}^2 = \frac{g^2}{1 - g^2 \bar{I}_2}$$

They kill infinities in the two-body sector.

- The known $\delta m_2, g_{02}$ are inserted in the two-body states of three-body sectors.
- In addition, there are $\delta m_3, g_{03}$ in the three-body states of three-body sectors.
- $\delta m_3, g_{03}$ are determined in from the remormalization conditions in three-body sector.
- They should kill infinities in the three-body sector.

• Renormalization condition

Reminder: $\bar{u}(k)\Gamma_2 u(p) = \bar{u}(k)\left[b_1 + \frac{m\not\omega}{\omega\cdot p} b_2\right] u(p)$

On energy shell $s = m^2$ we should impose:

1. $b_1(s = m^2) = g$ (relation between g_{03} and g)
2. $b_2(s = m^2) = 0$ (kills ω-dependence in Γ_2).
3. $M = m_{phys}$ (determines δm_3.)

To satisfy 2., we introduce the ω-dependent counter term by

$$g_{03} \to g_{03} + \frac{m\not\omega}{\omega\cdot p} Z_\omega$$

• x-dependent counter terms

However: $s = (k_1 + k_2)^2 = \dfrac{k_\perp^2 + \mu^2}{x} + \dfrac{k_\perp^2 + m^2}{1-x} = m^2$

This means $k_\perp^2 = -x^2 m^2 - (1-x)\mu^2 < 0$

(non-physical x -dependent value)

Renormalization condition:

$$b_1^{i=0,j=0}(g_{03}; k_\perp(x), x) = g, \quad k_\perp(x) = i\sqrt{x^2 m^2 - (1-x)\mu^2}$$

$b_1^{i=0,j=0}(g_{03}; k_\perp(x), x)$ depends on x because of truncation.

The same for the ω-dependent counter term: $Z_\omega = Z_\omega(x)$
to make $b_2^{i=0,j=0}(k_\perp, x) = 0$ at $s = m^2$, for any x.

Sector and x-dependent counter terms

St. Glazek, A. Harindranath, S. Pinsky, J. Shigemitsu, and K. Wilson, Phys. Rev. **D 47**, 1599 (1993).

In the initial Hamiltonian, the counter terms do not depend on the Fock sectors and kinematical variables.

Making truncation, we replace the initial Hamiltonian by a finite matrix.

The counter terms naturally depend on the dimension of matrix (sector dependence) and on kinematical variables (x-dependence).

Inspite of that, the counter terms are found absolutely unambiguously.

They (hopefully) provide finite results after non-perturbative renormalization.

• New components

A few technical details.

Introduce, for convenience:

$$
\begin{aligned}
b_1^{ij} &= \frac{m_i}{m} h_i^j, \\
b_2^{ij} &= \frac{m_i}{m} \frac{H_i^j - (1 - x + \frac{m_i}{m}) h_i^j}{2(1-x)}.
\end{aligned}
$$

• Renormalized equations

$$
\begin{aligned}
h_0^j(R_\perp,x) &= \eta g + g'^2\left[K_1^j h_0^j(R_\perp,x) + K_2^j h_1^j(R_\perp,x)\right] + g'^2 i_0^j(R_\perp,x),\\
h_1^j(R_\perp,x) &= g'^2\left[-K_3^j h_0^j(R_\perp,x) + K_4^j h_1^j(R_\perp,x)\right] + g'^2 i_1^j(R_\perp,x),\\
H_0^j(R_\perp,x) &= \eta g(2-x) + g'^2\left[K_1^j H_0^j(R_\perp,x) + K_2^j H_1^j(R_\perp,x)\right] + g'^2 I_0^j(R_\perp,x),\\
H_1^j(R_\perp,x) &= \eta g + g'^2\left[-K_3^j H_0^j(R_\perp,x) + K_4^j H_1^j(R_\perp,x)\right] + g'^2 I_1^j(R_\perp,x),
\end{aligned}
$$

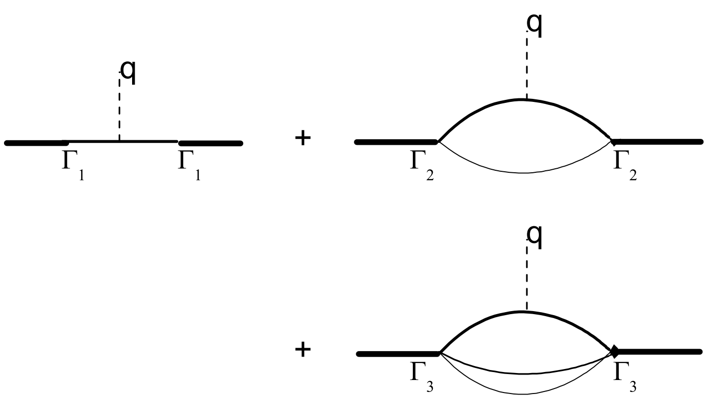
$i^j(R_\perp,x), I^j(R_\perp,x)$ are 2D integrals.

We solve them numerically (at laptop) and find the Fock components.

Knowing the Fock components (wave functions), we calculate e.m. form factors.

• EM form factors

1- and 2-body components are found from equations (model-dependent).
3-body components g_{1-4} are expressed through 2-body components (model-dependent).
Form-factors are expressed through 1-, 2- and 3-body components (model-independent).



1-, 2- and 3-body contributions in EM form factors

• Form factor F_1

V. A. KARMANOV, J.-F. MATHIOT, AND A. V. SMIRNOV PHYSICAL REVIEW D **86,** 085006 (2012)

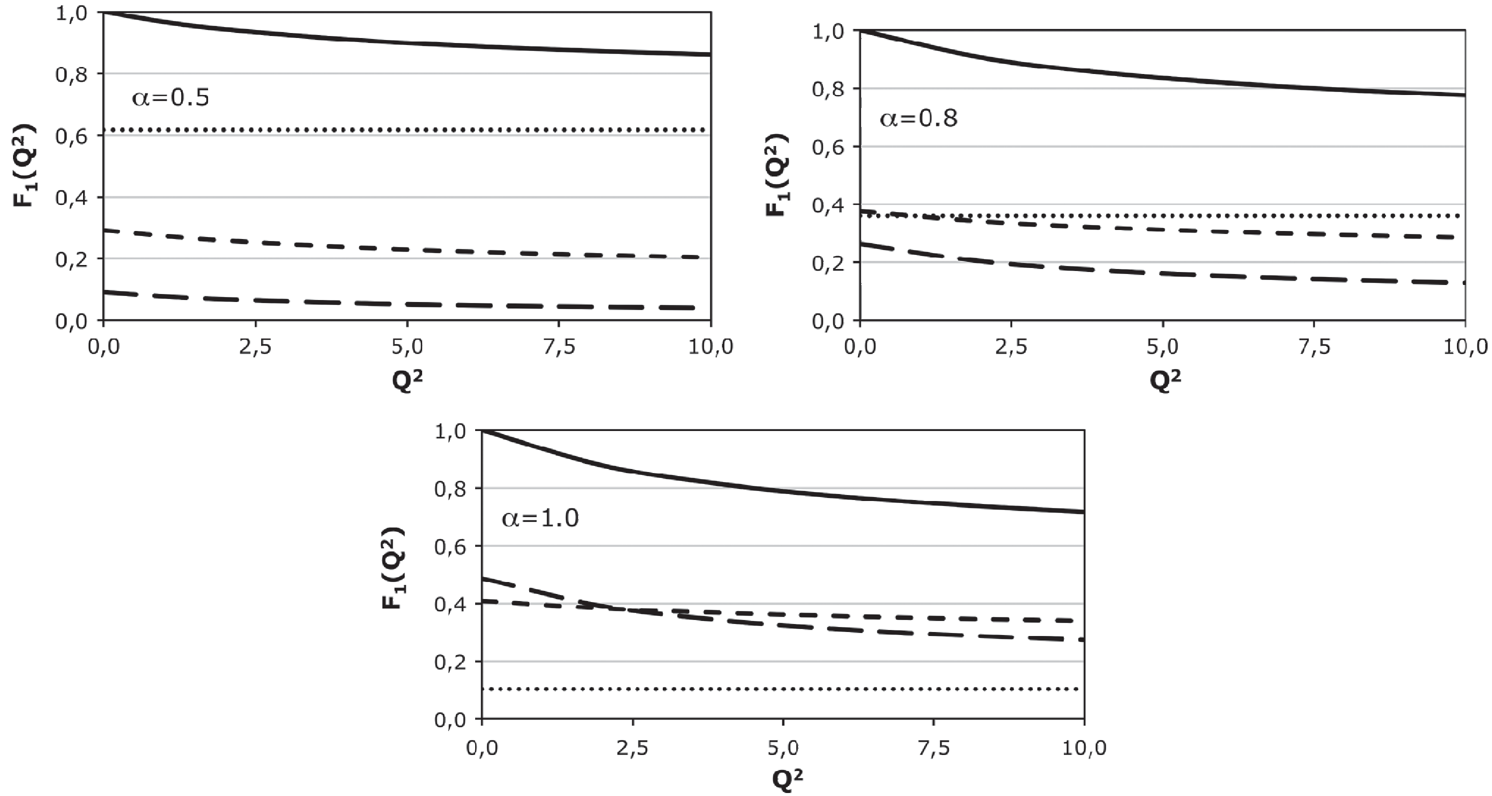


FIG. 7. Electromagnetic form factor $F_1(Q^2)$ in the Yukawa model, at $\mu_1 = 100$, for $\alpha = 0.5$ (upper left plot), 0.8 (upper right plot), and $\alpha = 1.0$ (lower plot). The dotted, dashed, and long-dashed lines are, respectively, the one-, two-, and three-body contributions, while the solid line is the total result.

Since h_i^j, H_i^j start growing from the characteristic values

The existence of a critical value for the regularization

• Anomalous magnetic moment

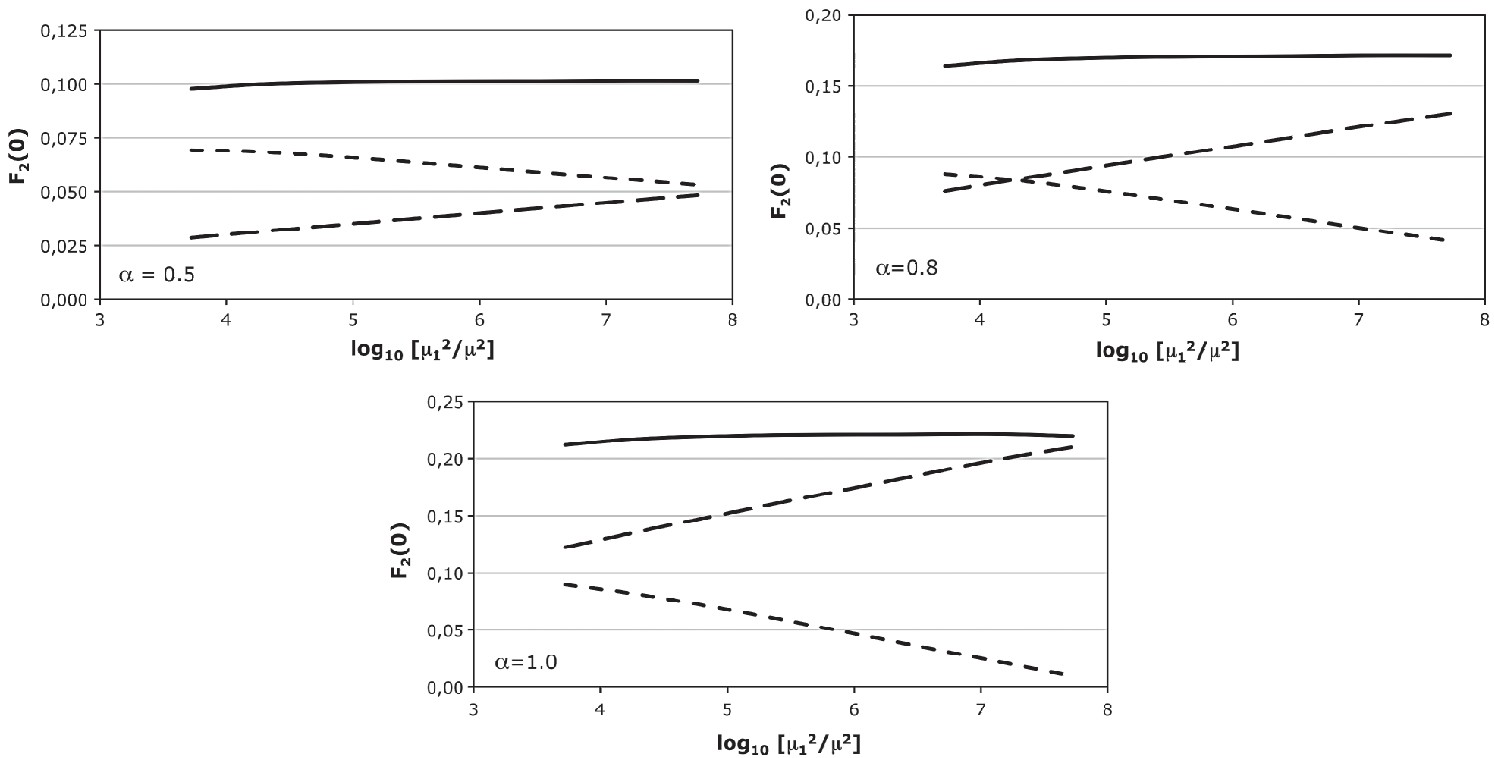


FIG. 6. The anomalous magnetic moment in the Yukawa model as a function of the PV mass μ_1, for three different values of the coupling constant, $\alpha = 0.5$ (upper left plot), 0.8 (upper right plot), and $\alpha = 1.0$ (lower plot). The dashed and long-dashed lines are, respectively, the two- and three-body contributions, while the solid line is the total result.

C. Numerical results

We finally show in Fig. 9 the contributions of the one-, two-, and three-body sectors to the norm of the state

• Adding antifermion ($ff\bar{f}$)

x-dependent counter term $Z_\omega(x)$

Dashed line – without $ff\bar{f}$. Solid line – with $ff\bar{f}$.

V. A. KARMANOV, J.-F. MATHIOT, AND A. V. SMIRNOV PHYSICAL REVIEW D **86**, 085006 (2012)

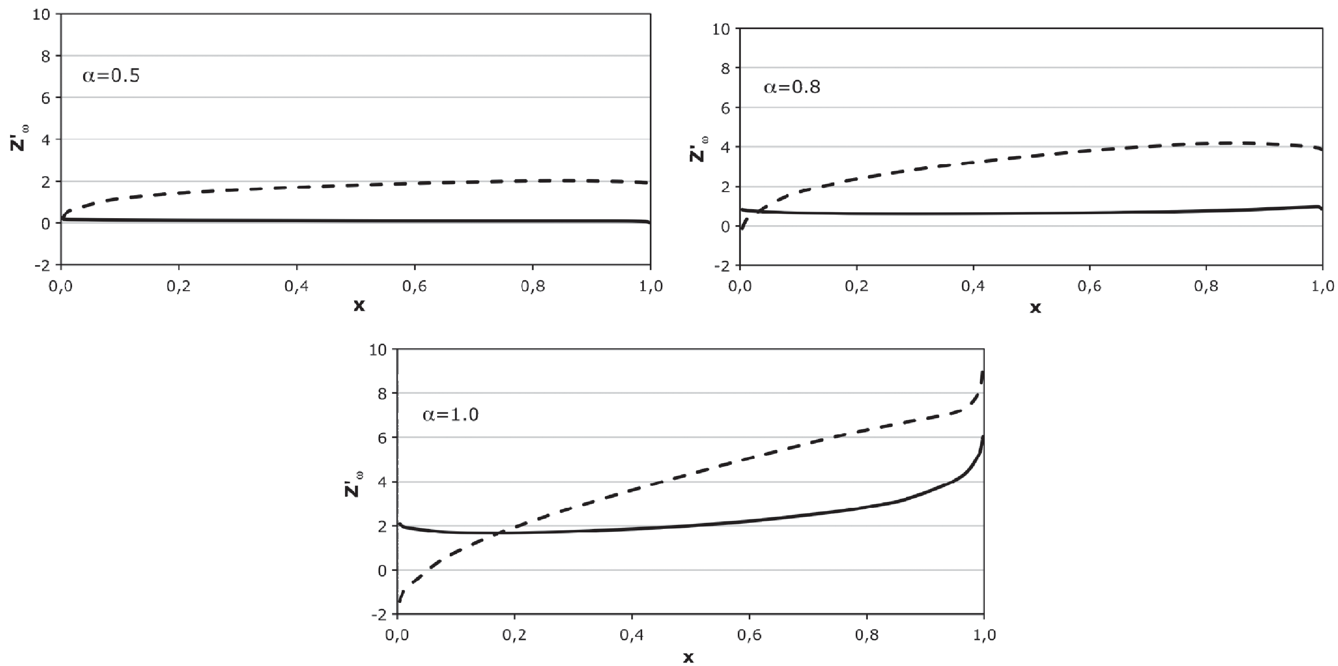
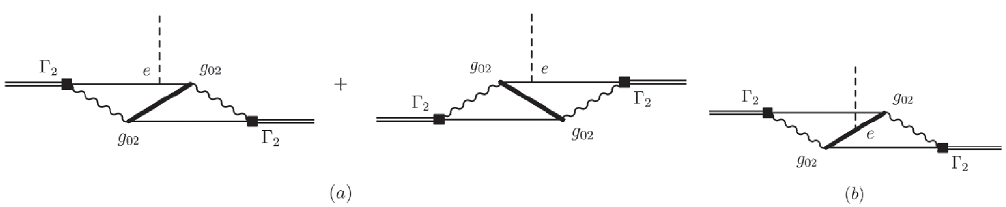
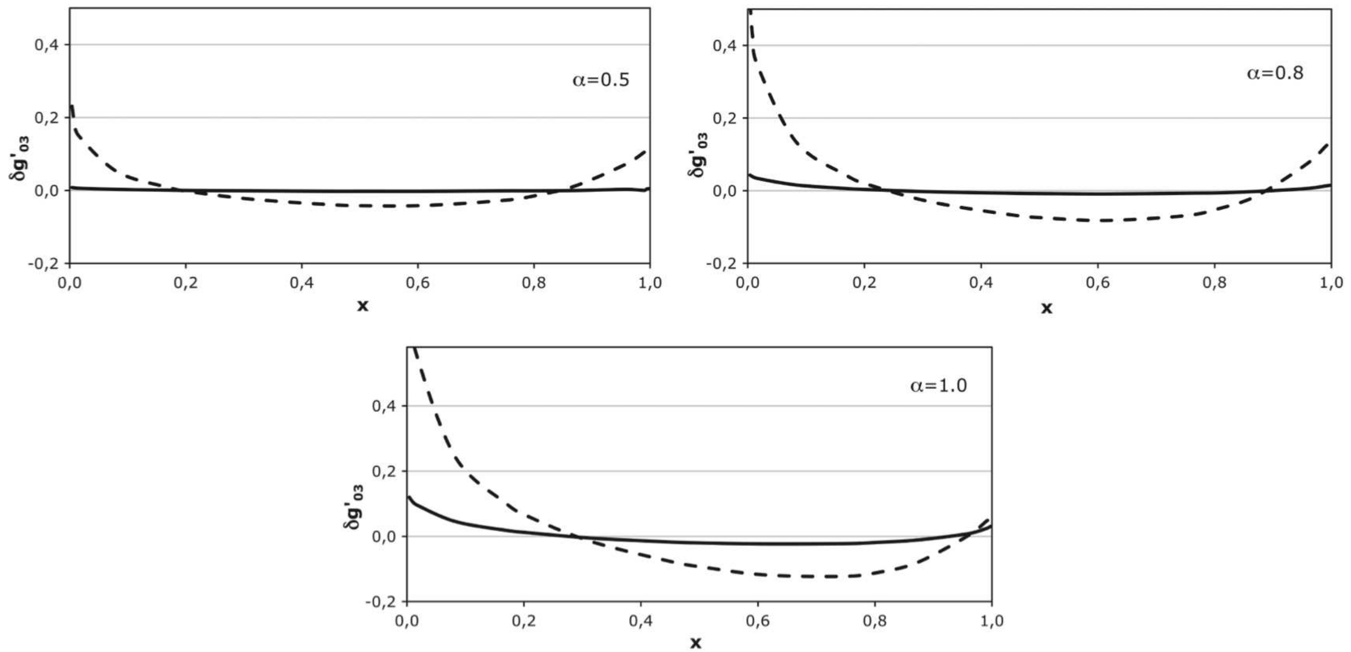


FIG. 14. x dependence of the counterterm Z'_ω for $\alpha = 0.5$ (upper left plot), $\alpha = 0.8$ (upper right plot), and $\alpha = 1.0$ (lower plot), calculated for $\mu_1 = 100$. The solid (dashed) lines correspond to the results obtained with (without) the $ff\bar{f}$ Fock sector contribution.



• Bare coupling constant $g_{03}(x)$

$$\delta g_{03}(x) = (g_{03}(x) - \bar{g}_{03})/\bar{g}_{03}$$



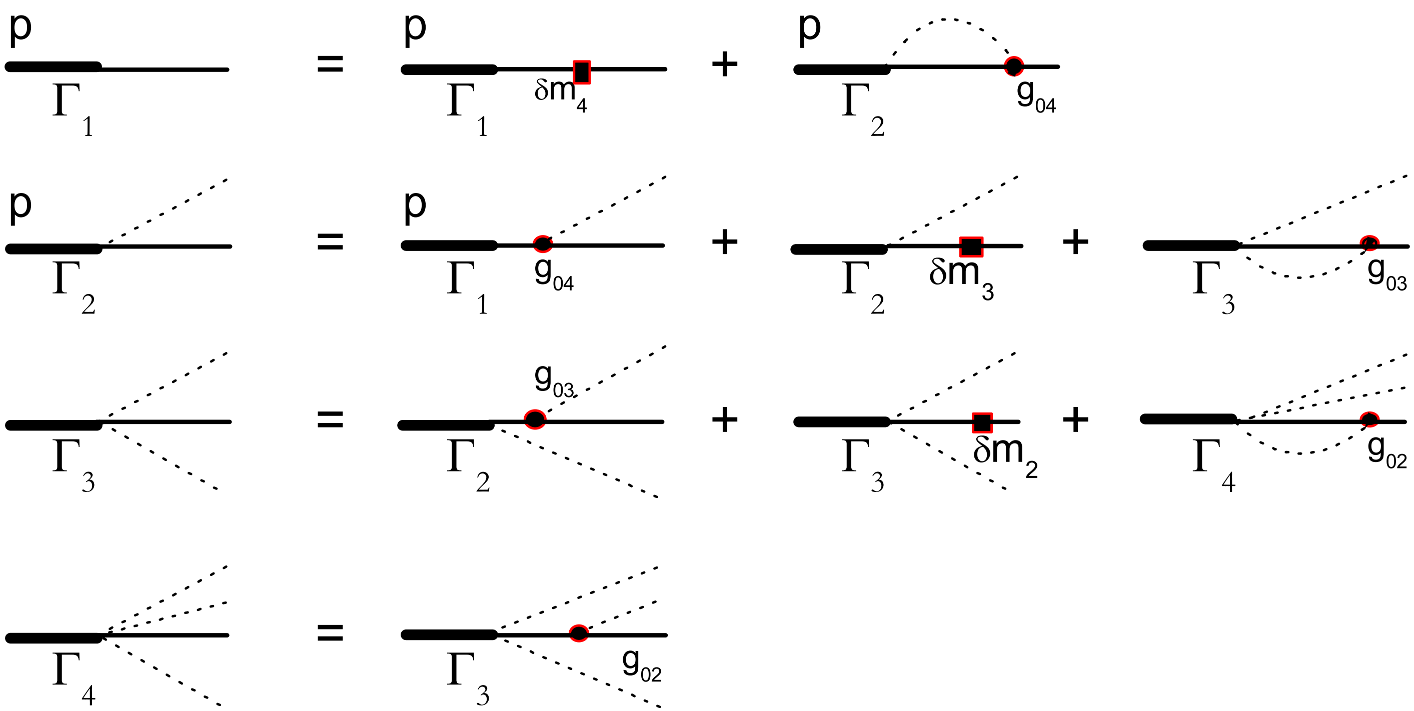
IG. 13. x dependence of the bare coupling constant g'_{03}, calculated relatively to its mean value over the interval $x \in [0, 1]$, for $= 0.5$ (upper left plot), $\alpha = 0.8$ (upper right plot), and $\alpha = 1.0$ (lower plot), calculated for $\mu_1 = 100$. The solid (dashed) lines orrespond to the results obtained with (without) the $ff\bar{f}$ Fock sector contribution.

Dashed line – without $ff\bar{f}$.   Solid line – with $ff\bar{f}$.

• Everything goes in good direction!

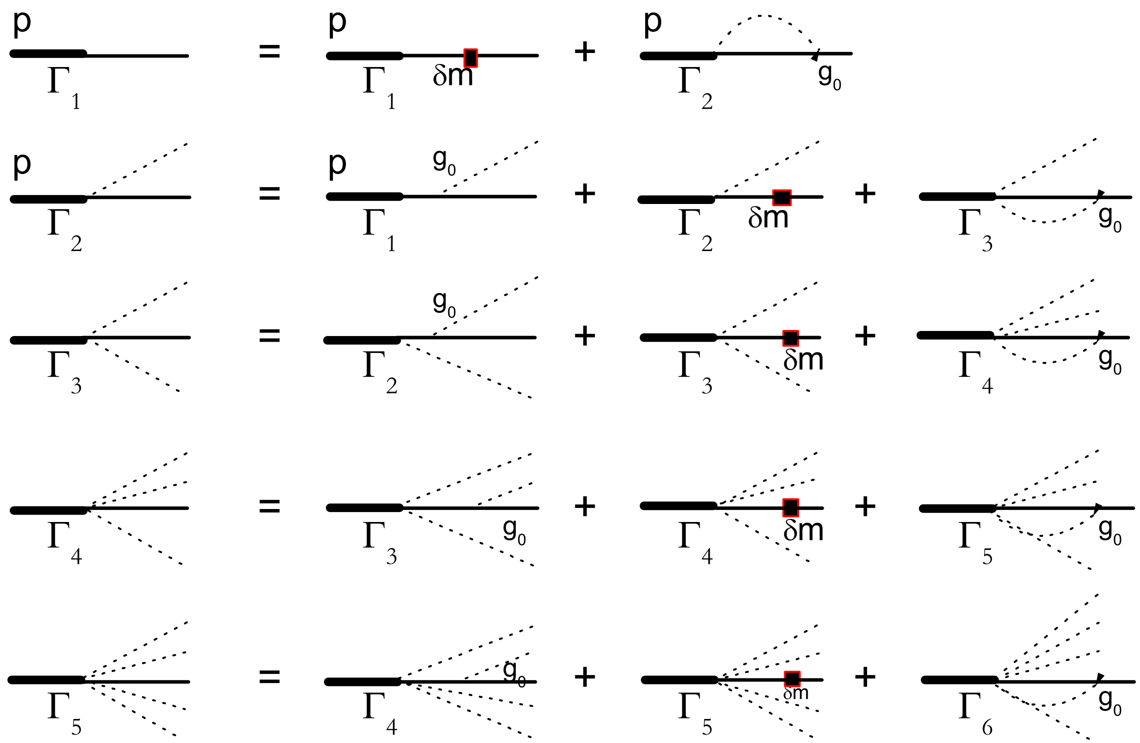
- Form factors do not depend on the PV masses when the latter tend to infinity – convergence.
- x-dependent counter terms become flat – stop to depend on x – when we increase the number of truncated states.

• Higher Fock sectors ($N = 4$)



System of equations for the vertex functions Γ_{1-4}.

• Higher Fock sectors ($N = 6$)



System of equations for the vertex functions Γ_{1-6}.

• Dimension of problem

$N = 2$ truncation.

Easily solved analytically.

$N = 3$ truncation

3-body wf is expressed via 2-body one $\psi_{ij}^{\sigma,\sigma'}(k_\perp, x)$.

16 values of indices i, j, σ, σ', 8 independent matrix elements.

Two variables $k_\perp, x$ (after separating azimutal angle).

We use spline basis: $\psi(k_\perp, x) = \sum_{j_1,j_2=0}^{2n+1} S_{j_1}(k_\perp) S_{j_2}(x) c_{j_1 j_2}$

$n = 5$ or 6 can give enough precision.

Dimension (for $n = 5$): $d = 2 \times 4 \times (2n+2)^2 = 8 \cdot 12^2 = 1152$.

Matrix $1152 \times 1152 \approx 1.3 \cdot 10^6$ elements. Solved at laptop.

Two body $N = 2$, three variables $\vec{k}_\perp, x = k_{\perp,x}, k_{\perp,y}, x$ (to avoid analytical angular integrals): $d = 2 \times 4 \times (2n+2)^3 = 13824$.

$N = 4$ truncation

4-body wf is expressed via 3-body one.

Three body, six variables $\vec{k}_{1,\perp}, x_1;\ \vec{k}_{2,\perp}, x_2$

Dimension (for $n = 5$):

$$d = 2 \times 4^2 \times (2n+2)^6 = 32 \cdot 12^6 \approx 0.95 \cdot 10^8.$$

Can be solved at supercomputer. We are solving it at ISU.

$$n = 4 \to d = 3.2 \cdot 10^7$$

$N = 5$ truncation

5-body wf is expressed via 4-body.

Four body, nine variables $\vec{k}_{1,\perp}, x_1;\ \vec{k}_{2,\perp}, x_2;\ \vec{k}_{3,\perp}, x_3$

Dimension (for $n = 6$): $d = 2 \times 4^3 \times (2n+2)^9 = 1.6 \cdot 10^{11}$.

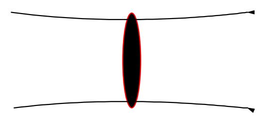
Can be hardly solved at supercomputer ...

• Remark

This counting is true but too straightforward.

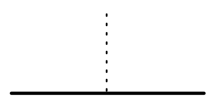
Being applied to the Schrödingier equation, it would "demonstrate" that one cannot solve more than three-body problem. But many body problem with large N is solved, in particular, at ISU (group of J. Vary).

Raison: simple two-body interaction.



The methods developed at ISU are successfully applied to nuclear theory. We can try to reformulate and apply these methods to field theory.

Raison: very simple basic interaction. All the Feynman graphs in their full complexity are made from these simple elements.



• Conclusion

- Non-perturbative approach, based on the truncation of Fock space, is developed.
- Approach is applied to the Yukawa model.
- Fock space is truncated up to three-body states, including state with antifermion ($ff\bar{f}$).
- E.M. form factors and anomalous magnetic moment are calculated.
- The results are stable (i.e., they converge) vs. increase of the meson PV mass.
- We should go to higher truncations.

• Resumé

Progress of the

Nuclear Theory in Supercomputer Era

opens exciting perspectives for a breakthrough in

the field theory.

This activity is inspired and supported by significant contribution of James Vary.

Happy Birthday, James!