Witten parameter and HPC on the lattice

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Confinement is the fundamental property of hadronic matter. The spectrum of the elementary particles can't be understood within QCD without this property. This feature results from string-like interaction between quarks at large distances. One of the possible explanation this interaction is condensation of magnetic monopoles in the vacuum[1]. According to t'Hooft[2], the monopoles can arise from the partial breaking of the gauge symmetry. In this work we consider $SU(2)$ symmetry breaking with conservation of $U(1)$ symmetry.

The surface operator[3] proposed by Witten[4,5] is a quantity sensitive to existence of monopoles. The parameter determine flow of the chromomagnetic field through a closed surface. All calculations are performed in the lattice approach with $SU(2)$ gluedynamics. We use multilevel and multi-hit algorithms on the supercomputer for obtain the best statistic.

In the Abelian vacuum (in electrodynamics) the flow vector of magnetic field through a closed surface is identically zero:

$$\oint \mathbf{H} \cdot d\mathbf{S} \equiv 0. \tag{1}$$

In the lattice calculation we use phase $e^{\imath\varphi}$, therefore the identity (1) acquires the next form:

$$e^{\imath\kappa \oint \mathbf{H}\cdot d\mathbf{S}} \equiv 1, \tag{2}$$

where κ is dimensional-eliminate coefficient. This identity holds in the Abelian theory with continuous simply connected space. If space-time have non-trivial topology or group symmetry is non-Abelian, then identity (2) does not necessarily hold. Then, in general case lattice quantum chromodynamics has:

$$e^{\imath\kappa\sum\limits_k \mathbf{H}_k\cdot\Delta\mathbf{S}_k} \neq 1, \tag{3}$$

where $\mathbf{H}_k$ - vector of magnetic field on the k-th lattice plaquette, $\Delta\mathbf{S}_k$ - area of the his surface (with the normal vector to the center of the plaquette), and integral is calculated through the closed surface. Closed surface is compiled by the lattice plaquettes.

Thus, we will call Witten parameter the following value:

$$W(S) = Re \prod_S e^{\imath\theta_p}, \tag{4}$$

Here θ_p is plaquette angle and S is surface. This parameter senses the chromomagnetic field. Phase is changed in this value of angle when moving along the contour of the plaquette. This phase is related with magnetic field flow through plaquette surface:

$$\kappa \int\limits_S \mathbf{H} \cdot d\mathbf{S} = \kappa \int rot\mathbf{A} \cdot d\mathbf{S} = \kappa \oint \mathbf{A} \cdot d\mathbf{l} = \theta_p, \tag{5}$$

where integration over $d\mathbf{l}$ is carried out on the path covering the surface S.

We can rewrite magnetic field flow:

$$\int\limits_S \mathbf{H} \cdot d\mathbf{S} = \int\limits_S F_{ik}\, d\sigma_{ik}, \tag{6}$$

where F_{ik} - the gauge field tensor, $d\sigma_{ik}$ - surface element (here we do not distinguish between upper and lower indices, because all calculations are performed in Euclidean space-time after Wick rotation), i, $k = 1, 2, 3$ - space direction. In this work we consider gauge field theory without particle with $SU(2)$ group symmetry broken up to $U(1)$ group symmetry. Thus θ_p is related with $F_{\mu\nu}$ by the following formula:

$$F_p = \widehat{1}\, cos\,\theta_p + \imath\, n_i\sigma_i\, sin\,\theta_p, \tag{7}$$

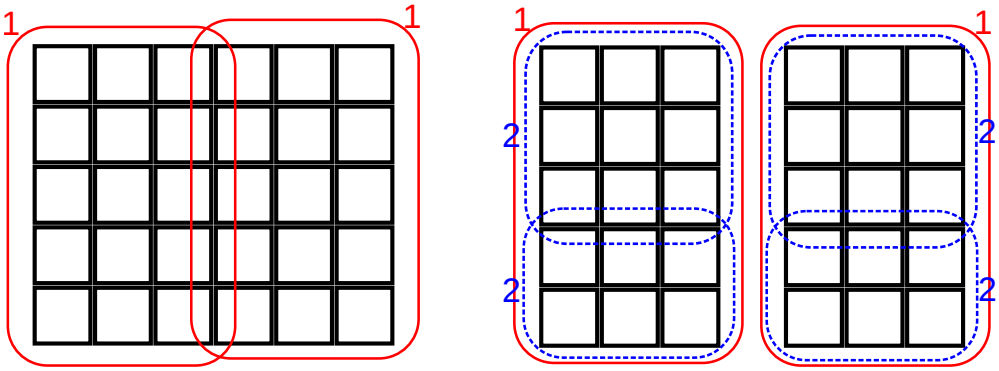

FIG. 1: Each planes are divided into two pieces on each level of recursion. 1 shows, that phase on the area (circled by red solid line) is calculated on the first level of recursion. 2 shows, that phase on the area (circled by blue dashed line) is calculated on the second level of recursion.

where n_i - vector on the unit sphere, σ_i - Pauli matrices, F_p is the value of the gauge field tensor $F_{\mu\nu}$ on the plaquette. Then, for the θ_p we can write follow definition:

$$\theta_p = \arccos\left(\frac{1}{2} Tr\, F_p\right). \tag{8}$$

The range of function $\arccos(x)$ is $[0, \pi]$. In the gauge group $U(1)$ the range of variation of the angle is $[0, 2\pi]$. Thereby, symmetry is broken to $U(1)$ up to a random phase $\varphi \in Z_2$.

Witten parameter isn't only relate with magnetic field, also it determines the correlation length for average plaquette:

$$\langle P \rangle = \left\langle 1 - \frac{1}{2} Tr\, U_p \right\rangle, \tag{9}$$

where U_p is plaquette variable, defined by $U_p = U_{ij} U_{jk} U_{kl} U_{li}$, where U_{ij}, U_{jk}, U_{kl}, U_{li} is plaquette edges. The distance between plaquettes (located far from each other) become larger with increase the surface. Thus statistically association between plaquetts becoming less. In this case the Witten parameter must vanish with an increase in area of the closed surface.

In the lattice approach we select cube in the 3d space with some length of the edge on the lattice. Anchor point is the left point in all axes on the cubic. The phase on the surface of the cube is calculated on the each plaquetts and result is obtained by summation these phases. Of course we remember sign of phase, it determined by normal vector to the plane of cube. After then we calculate our parameter at different point in the lattice configuration and average them. The final result is obtained by averaging on the set of configurations.

We consider cubic with length of the edge from $1a$ to $13a$ (a is the lattice scale), which corresponds surface from 6 to 1014 plaquetts. For best accuracy we use multilevel[6], multi-hit[7] algorithms and MPI (Message Passing Interface) parallelism for fast calculation.

Cubic we can separate into 6 plane. Thus, we need to learn how phase is calculated on this plane. The recursive function was written which uses multilevel scheme and makes following:

1. If depth of recursive above some parameter $depth_{max}$ or plane is one plaquette then phase on the current plane is calculated by multi-hit algorithm and return this value.

2. Current plane is divided into two pieces by perpendicular to the axis. The length of the plane edge is maximum along this axis, see Figure 1.

3. This function is recursively called for each pieces on the plane and calculate phase on this plane φ_i (φ_i is sum phases on the pieces of plane).

4. When we accumulate enough quantity of φ_i, then averaging them and return result. Else function run some quantity Monte Carlo passages on the current plane and go to step 2.

This function was tested. We select different value of depth, from 1 to 3 and calculate dependence Witten parameter from surface area on some set of lattice configuration. We obtained that multilevel algorithm is well converges. Result of calculations for recursive depth level equal two and three are close. For all calculation in this work value of depth is equal three.

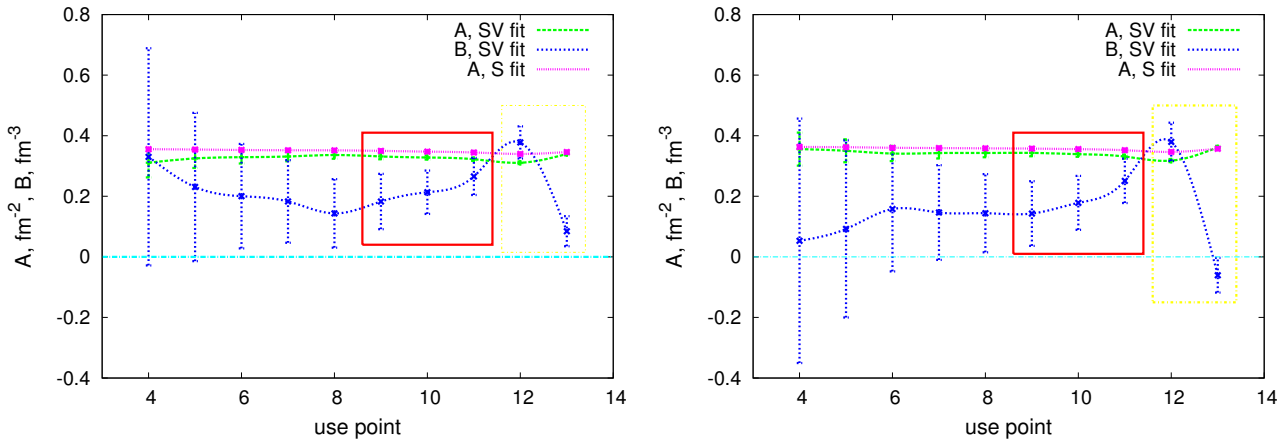

FIG. 2: Witten parameter fitting in the confinement phase (leftward) and in the deconfinement phase (rightward). Where A - surface coefficient, B - volume coefficient, SV fit - fitting with use dependence from area surface and volume, S fit - use dependence only from area surface. Red solid rectangle is the best area for get results.

It is the main idea of the multi-hit algorithm that phase on the plaquette is defined by bound conditions. We can substitute phase calculated on the plaquette to phase expressed by bound links. We can't do it analytically, but we can use Monte Carlo algorithm for obtain more accurate phase value. For this purpose, we calculate set of phase on the one plaquette. Between calculating we run Monte Carlo algorithm on the plaquete. After that the final phase was obtained by averaging set of phase.

Using this two algorithms in the calculations gives best accuracy for Witten parameter, but work time increases. Therefore we use MPI in our calculation. Parallelism is achieved by calculating Witten parameter at the same time in the different points on the lattice. Each supercomputer core calculate our parameter in some points on the lattice configuration. We use 144 cores for each phase.

All calculation performed on 50 configurations in 1000 points on the each lattice configuration. The behaviour of Witten parameter is fitted by next dependence:

$$W(S) = e^{-AS}, \tag{10}$$

$$W(S,V) = e^{-AS-BV}, \tag{11}$$

where A is surface coefficient, B is volume coefficient, S is area surface, V is volume of cube.

For small surface we have good statistic and bad approximation. Therefore we studied dependence of coefficient A, B from quantity use point in fitting. We discard n first point for small surface and the remaining points are fitted by dependence (10) and (11). Figure 2 is shown that. Small quantity of use point makes big dispersion for coefficient A, B, thus we can't use them in analysis. The most appropriate area for conclusion is singled out by red solid rectangle. It turns, that Witten parameter depend from area surface and volume in two phase. Consequently it isn't the order parameter for phase transition confinement-deconfinement.

The result is multilevel algorithm stated on the close surface. This algorithm is tested on the Witten parameter calculating. Accuracy increased by two orders of magnitude with respect calculating without the use of the algorithm.

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